Introduction to Differential Geometry

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Problems in differential calculus.

Method of reduction to a special case.

Suppose we would like to prove some statement S. Often we can find some statement S_0 which is a special case of S and prove that if S_0 is always true then S is always true.

Now in order to prove S it is enough to prove S_0 which might be easier since this is a special case (some special situation). Often the simplification comes from the fact that in this case notations are simpler.

Here are some examples.

1. Statement S. Let U be a domain in $\mathbf{R}^n$, f a function on U. Suppose we know that all partial derivatives $f_i = \partial_i f$ exist at all points of U and are continuous functions. Then f is differentiable at any point $a \in U$.

(i) Show how to reduce S to the special case $a = 0$.

(ii) Show how to reduce S to the special case $a = 0$, $f(a) = 0$.

[**P**] (iii) Show how to reduce S to the special case $a = 0$, $f(0) = 0$, $f_i(0) = 0$ for all i.

[**P**] (iv) Prove that the special case (iii) of the statement S follows from the following

Bounding Lemma. Let f be a function on the open ball $B = B(0, r) \subset \mathbf{R}^n$. Suppose we know that $f(0) = 0$, all partial derivatives $f_i(x)$ exist and are bounded by some constant A for all $i = 1, ..., n$ and all $x \in B$. Then $|f(x) \leq nA||x||$ for all $x \in B$.

(v) Prove the lemma in (iv) and hence the statement S.

2. Statement S. Let $\phi : U \to V, \psi : V \to W$ be maps of domains. Consider a point $a \in U$ and set $b = \phi(a) \in V, c = \psi(b) \in W$.

Suppose we know that phi is differentiable at the point a and with differential A and ψ is differentiable at the point b with differential B.

Then the composition morphism $\psi \cdot \phi$ is differentiable at the point a and its differential equals BA.

(i) Reduce this to the case $a = 0, b = 0, c = 0$.

[**P**] (ii) Show that this case easily follows from the fact that the estimate $||\psi(y)|| \leq C||y||$ holds for all points y inside some ball $B(0, r)$.

[**P**] **3.** Statement S. Let f be a function on domain U with coordinates x_i. Suppose we found two indexes $i \neq j$ such that the mixed partial derivatives f_{ij} and f_{ji} exist everywhere in U and are continuous at a point $a \in U$. Then they are equal at this point.

(i) Reduce to the case when U is a ball on the plane with two coordinates x, y centered at the point $a = 0$ and the function $f = f(x, y)$ vanishes on the cross (i.e. $f(x, 0) \equiv 0$ and $f(0, x) \equiv 0$).

(ii) Deduce the last case from

Lemma. Let f be a function in a ball B around 0 in the plane (with coordinates x, y). Suppose that f vanishes on the cross and the mixed partial derivative f_{xy} exists everywhere and is bounded by A. Then $|f(x, y)| \leq A(x^2 + y^2)$.