Highlights of the Lectures

Introduction to Differential Geometry.

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One of the most central ideas of Analysis and of Differential Geometry is the idea of linearization. Namely the idea is that we start with "analytical" problem, which in some sense is "difficult", and try to approximate it with a "linear" problem. Then we solve this linear problem, and this gives us an important information about the original problem.
The attitude here is that any linear problem should be somehow "elementary" (trivial).
In order to employ this general idea one should really feel that linear problems are elementary, which means that one should be well familiar with linear algebra.

So I start with a brief exposition of basic notions of linear algebra. You probably have seen most of these things before, but since I will use them all the time in a very significant way I prefer to repeat them.
Also for me it is important that I will be working in invariant - i.e. coordinate free - way. This means that I will try to give definitions and prove results without introducing coordinates but then will try to illustrate how all this looks if we write everything in terms of coordinates.
Later we will see that the same scheme works in the study of manifolds.

I. Basic notions of Linear Algebra.

1. Definition of a vector space (over $\mathbf{R}$).
By definition a vector space is a set V equipped with two operations:

Addition $V \times V \to V$, notation $(x, y) \mapsto x + y$
Multiplication $\mathbf{R} \times V \to V$, notation $(a, x) \mapsto ax$

satisfying some axioms of vector space.

If V is a vector space, then a subspace of V is a subset $W \subset V$ which contains 0 and is closed with with respect to addition and multiplication.

Basic examples.
(i) Space $\mathbf{R}^n$.
(ii) Let I be the segment $I = [0, 1]$. Consider the space $V = \mathcal{F}(I)$ of all functions on I. It has subspaces $C(X)$ - continuous functions, $C^\infty(I)$ -smooth functions, $Pol(I)$ - polynomial functions, $Pol(I)^{\leq k}$ - polynomial functions of degree $\leq k$ and so on.

2. Definition of a morphism (linear operator) $A : V \to L$.

If $A : V \to L$ is a linear operator , we define its **kernel** $K = \ker A$ as $K = \{x \in V | Ax = 0\} \subset V$ and its image $I = A(V) \subset L$. These are subspaces of V and L.

3. Let $v_1, ...v_n$ be a collection of vectors in V. We denote by $< v_1, ..., v_n >$ the **span** of these vectors, i.e. the minimal subspace containing these vectors. It is easy to see that this subspace consists of all linear combinations of vectors v_i, i.e. of vectors v of the form $\sum a^i v_i$ with $a^i \in \mathbf{R}$.

We say that vectors $v_1, ..., v_n$ are **linearly independent** if any non-trivial linear combination of them is not 0. This means that linear combinations with different coefficients describe different vectors in V.

We say that vectors $v_1, ..., v_n$ form a **basis** of the space V if they are linearly independent and span the whole space V.
This means that any vector $v \in V$ can be written as a linear combination $v = \sum a^i v_i$ in a unique way. Hence the coefficients of this linear combination can be considered as functions of v. The function $v \mapsto a^i = a^i(v)$ is called **the i-th coordinate with respect to the basis** (v_i).

4. **Definition**. A space V is called finite-dimensional if there exists a finite collection of vectors $v_1, ..., v_n \in V$ which span V.

The first basic result of linear algebra is the following

Theorem. Let $f_1, ..., f_m$ and $e_1, ..., e_n$ be two collections of vectors in V. Suppose we know that the vectors (f_i) are linearly independent and vectors (e_j) span V.

Then $m \leq n$.

Corollary. Let V be a finite dimensional vector space. Then it has a basis. The number of elements in this basis does not depend on the choice of this basis.

The number of elements in any basis of V is called the **dimension** of V, notation $\dim V$.

5. Important remark. Let V be an n-dimensional space. Then we can choose a basis $e_1, ..., e_n$ of V and using the corresponding coordinates a^i we construct an isomorphism $a : V \to \mathbf{R}^n, v \mapsto (a^i(v))$.
Thus we can realize V as the space $\mathbf{R}^n$.
But this realization depends on the choice of the basis. Usually there is no special basis which is better, or more canonical, than other basses. Thus this realization is not canonical and as a result many properties valid in one realization will not be valid in others.
The general idea is that we would like to study some facts (properties) which do not depend on realization (choice of a coordinate system). After we found such property we can write how it looks in a particular coordinate system

6.. Constructions of new vector spaces from old ones.

(i) **Dual space.** Let V be a vector space. We denote by V^* the set of all linear functionals $f : V \to \mathbf{R}$. This set has a natural structure of a vector space which is called **vector space dual to V**.
If $e_1, ..., e_n$ is a basis of V then the corresponding coordinate functions f^i form a basis of the dual space V^*; it is called the **dual basis.**

It is easy to see that in this case the double dual space $(V^*)^*$ is **canonically** isomorphic to V.

(ii) More generally, given vector spaces V, L we can consider the vector space $M = Mor(V, L)$ of linear operators from V to L.
If $e_1, ..., e_n$ is a basis of V, $f_1, ..., f_m$ basis of L then every operator $A : V \to L$ can be described by coefficients A_i^j such that $A(e_i) = \sum A_i^j f_j$.
The functions $A \mapsto A_i^j$ form a coordinate system on the vector space M. In other words, the choice of coordinates on V and L allows us to realize the vector space M, which was constructed without any coordinates, as the space of $m \times n$-matrices.

Excercise.Show that the product of operators corresponds to the standard product of matrices.

(iii) **Quotient space.** Let $W \subset V$ be a subspace. Then there exists a vector space Q and a linear operator (projection) $p : V \to Q$ such that p is epimorphic and $\ker p = W$.
The pair (Q, p) is defined absolutely canonically, i.e. any two such pairs are canonically isomorphic. The space Q is usually denoted by V/W and is called the **quotient** space.