Problem assignment 4.

Introduction to Differential Geometry.

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[P] 1. Let X be a metric space, $C(X) \subset \mathcal{F}(X)$ be the subalgebra of all continuous functions.

(i) Suppose we know only the set X and the subalgebra $C(X) \subset \mathcal{F}(X)$. Show how to reconstruct the topology on X.

(ii) Let X, Y be two metric spaces and $\pi : X \to Y$ be any map of sets.

Show that π is continuous iff $\pi^*(C(Y)) \subset C(X)$.

Show that in this case $\pi^* : C(Y) \to C(X)$ is a homomorphism of $\mathbf{R}$ algebras, and that this homomorphism π^* uniquely determines the map π.

($\square$)**2.**. We say that a metric space is finite dimensional if it is homeomorphic to a subset of an Euclidean space $\mathbf{R}^N$ for some N.

Suppose that in the situation of problem 1 (ii) the space Y is finite dimensional. Show that any homomorphism of $\mathbf{R}$-algebras $\nu : C(Y) \to C(X)$ comes from some (uniquely defined) continuous map $\pi : X \to Y$. (Here homomorphism ν is supposed to be linear, preserve multiplication and the element 1).

In other words, for such spaces the algebra $C(X)$, considered as an abstract $\mathbf{R}$-algebra, completely determines the topological space X.

Hint. Consider first the case when X is a point and $Y = \mathbf{R}^n$.

(ii) Suppose in the situation of problem 1 (ii) the space Y is compact. Prove the same statements that in part (i).

3. (i) Let X be a domain, $a \in X$. Let us consider two coordinate systems (x^i) and (y^j) on X and denote by d_x and d_y the corresponding metrics on the space X.

Show that there exists a neighborhood W of the point a on which these two metrics are comparable.

Let γ_1, γ_2 be two short curves at the point a.

Show that these curves are equivalent iff $d(\gamma_1(t), \gamma_2(t)) = o(t)$, where d is the standard Euclidean distance with respect to some coordinate system on X.

[P] 4. Extension property for functions on manifolds. Let M be a manifold (in our case just a submanifold in some $\mathbf{R}^N$), a a point of M and W a neighborhood of a in M.

Show that there exists a smooth function $u \in S(M)$ such that $u(x) = 0$ for points x outside of W and $u(x) \equiv 1$ for points x close to a.

Using this show that for any smooth function $f \in S(W)$ we can find a smooth function $h \in S(M)$ such that $f \equiv h$ near the point a.

[P] 5. Let M be a manifold, $S(M)$ the algebra of smooth functions on M. For a point $a \in M$ we define the tangent space $T_a(M)$ to be $T_a(W)$ where W is some open smooth domain containing a.

(i) Show that the natural morphism $T_a(M) \to Der_a(S(M))$ is an isomorphism.

(ii) Let $Vect(M)$ denote the space of (smooth) vector fields on M.

Consider the natural morphism $i : Vect(M) \to Der(S(M))$ of the space of smooth vector fields on M to the space of derivations of the algebra $S(X)$ given by $i(\xi)(f) = \xi(f)$.

Show that this is an isomorphism of vector spaces.

(**Hint.** Use problem 4).

[P] 6. Let (X, S(X)) be an abstract smooth domain. Suppose we know only the set X and the algebra of functions $S(X) \subset \mathcal{F}(X)$.

(i) Show how to reconstruct the topology on X.

For every open subset $U \subset X$ show how to reconstruct the algebra $S(U)$.

(ii) Show that for two smooth domains X, Y a map of sets $\pi : X \to Y$ is smooth iff $\pi^*(S(Y)) \subset S(X)$. Show that in this case the morphism of algebras $\pi^* : S(Y) \to S(X)$ completely determines morphism π.

(□)**7.** Show that a smooth domain $(X, S(X))$ can be completely reconstructed from abstract algebra $S(X)$. In particular, for two smooth domains X, Y any morphism of $\mathbf{R}$-algebras $\nu : S(Y) \to S(X)$ comes from (unique) smooth map of smooth domains.

8. Let A be an associative algebra (for example, algebra of endomorphisms of some linear space V). Let us define a new operation $[\ ,\ ]$ on the vector space A by $[a, b] = ab - ba$ (this operation is called **commutator**).

(i) Show that the commutator $[,]$ is a bilinear skew-symmetric operation. Show that it satisfies Jacobi identity

$$[a, [b, c]] + [c, [a, b]] + [b, [c, a] = 0$$

(ii) Fix an element $a \in A$ and consider the operator $D = Ad_a : A \to A$ given by $D(x) = [a, x]$. Show that D is a derivation for both multiplication operation $(a, b) \mapsto ab$ and commutator operation $(a, b) \mapsto [a, b]$.

9. For an arbitrary algebra A check that the subspace of derivations $L = Der(A) \subset Op(A)$ is closed with respect to commutator.

Show that the commutator operation on the space L is skew-symmetric and satisfies the Jacobi identity.

Note. By definition this condition means that the space L with commutator operation is a **Lie algebra**.