Problem assignment 5.

Introduction to Differential Geometry.

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1. Check that the multiplication in the algebra $Alt(V)$ is associative and super commutative.

2. Check that the DeRham differential d given by $d(f dx_{i_1}...dx_{i_k}) = df dx_{i_1}...dx_{i_k}$ satisfies super Leibnitz rule.

[P] 3. Consider the space of matrices $Y = Mat(n, \mathbf{R})$ as a manifold.

(i) Let $X = Y \times Y$ and $m : X \to Y$ the morphism defined by multiplication, $m(A, B) = AB$.

Compute differential Dm at the point $a = (A, B) \in X$.

(ii) Consider morphism $\nu : Y \to Y$ defined by $A \mapsto A^5$. Compute the differential $D\nu$ at every point.

[P] 4. Let $X = \mathbf{R}^3$ and $S \subset X$ be the unit sphere.

(i) At every point $a \in S$ describe the tangent space $T_a S$ as a subspace of $T_a X = \mathbf{R}^3$. Describe the Riemannian metric on S induced by the standard Riemannian metric on X.

(ii) Let f be a smooth function on X and h its restriction to S. Then we can construct two vectors $\xi = grad(f) \in T_a X$ and $\eta = grad(h) \in T_a S$.

Show that the vector η is the orthogonal projection of the vector ξ.

Show that the same result holds for any submanifold $S \subset X$.

[P] 5. Let $X = \mathbf{R}^3$. Consider the following subsets of X given by equations

S given by equation $x^2+y^2+z^2 = 1$, H given by equation $x^2+y^2-z^2 = 1$, R given by equation $x^3+y^3+z^3 = 1$ and C given by equation $x^2+y^2-z^2 = 0$.

(i) Show that S, H, R are submanifolds. Describe their tangent spaces at all points.

(ii) Show that C is a submanifold at all points except 0.

(iii) Show that the subset $Z = S \bigcap R$ is a submanifold. Describe its tangent spaces at all points.

6. Let R be a subset of a topological space X. The subset R is called **locally closed** if every point $a \in R$ has an open neighborhood W in X such that $R \bigcap W$ is closed in W.

Show that R is locally closed iff there exists an open neighborhood U of R in X such that R is closed in U.

[P] 7. Let X be a domain and $C \subset X$ be a compact subset.

(i) Show that there exists an open neighborhood U of C in X which lies inside a compact subset $K \subset X$.

(*/2) (ii) Show that there exist sequences of compact subsets C_i and open subsets U_i of X such that $C = C_1 \subset U_1 \subset C_2 \subset U_2, ...$ and $\bigcup U_i = X$.

(□)**8.** Prove the following generalization of the inverse function theorem (it is called **constant rank theorem**).

Theorem. Let $\pi : X \to Y$ be a morphism of smooth domains, $a \in X$. Suppose we know that the differential $D\pi_x : T_x X \to T_{\pi(x)} Y$ has constant rank k for every point $x \in X$.

Show that one can find coordinates $(x^1, ..., x^m)$ near point a and $(y^1, ..., y^n)$ near the point $b = \pi(a)$ in which the morphism π takes the standard form $\pi(x^1, ..., x^m) = (x^1, ..., x^k, 0, 0, ..., 0)$.

[P] 9. Consider on $\mathbf{R}^2$ the 1-form $\alpha = xdy$.

(i) Compute the integral of the form α over the unit circle S.

(ii) Consider the integral of the form α over the curve γ given by $x > 0, y > 0, x^3 + y^3 = 1$.

Write down how to evaluate this integral (as an integral in one variable).

10. Let B be a symmetric $n \times n$ matrix of real numbers. Consider its characteristic polynomial $P(t) = \det(t1_n - B)$.

Show that all the roots of the polynomial P are real numbers.