Problem assignment 4.

Representations of Finite Groups.

Joseph Bernstein March 22, 2007.

In this assignment we fix a field k. For every set X we denote by $Sh(X) = Sh(X, k)$ the category of sheaves of k-vector spaces over X.

1. Let $\pi : X \to Y$ be a map of sets. Define the **" inverse image"** functor $\pi^* : Sh(Y) \to Sh(X)$ by $\pi^*(\mathcal{H})_x = \mathcal{H}_{\pi(x)}$.

(i) Describe how this functor acts on morphisms.

(ii) Show that the functor π^* has right adjoint functor $\pi_* : Sh(X) \to Sh(Y)$ (it is called the **direct image** functor. Describe explicitly the action of this functor on objects and morphisms.

(ii) Show that the functor π^* has left adjoint functor $\pi_! : Sh(X) \to Sh(Y)$ (sometimes it is called the proper direct image functor).

Describe explicitly the action of this functor on objects and morphisms.

Show that there exists a canonical morphism of functors $\pi_! \to \pi_*$ which is an isomorphism if the sets X, Y are finite.

2. Fix a group G. For every G-set X we denote by $Sh_G(X) = Sh_G(X, k)$ the category of G-equivariant sheaves of k-vector spaces over X.

For any morphism of G sets $\pi : X \to Y$ describe the functor $\pi^* : Sh_G(Y) \to Sh_G(X)$ and the adjoint functors π_* and $\pi_!$.

3. Describe sums and products of objects in the categories $Sh_G(X)$.

[P] 4. (i) Let X be a free G-set. Show that the category $Sh_G(X)$ is **canonically** equivalent to the category $Sh(\bar{X})$, where $\bar{X} = G\backslash X$.

If you have done this correctly you should be able to deduce from this the following more general statement:

(ii) Let R be a group which contains G as a normal subgroup. Consider the quotient group $Q = R/G$.

Suppose we have an R-set X that is free as a G-set. Show that the category $Sh_R(X)$ is canonically equivalent to the category $Sh_Q(\bar{X})$.

[P] 5. Let $GSets$ denote the category of G-sets and $GTors$ denote the full subcategory of G-torsers in the category $GSets$.

Show that the category $Sh_G(pt) = Rep(G)$ of representations of G over a field k is canonically equivalent to the category of functors $Funct(GTors \to Vect_k)$ from the category $GTors$ to the category of k-vector spaces.

Definition. If C is any category and Z an object of C we define a category C_Z as follows:

$Ob(C_Z)$ consists of pairs $(X \in Ob(C), p_X : X \to Z)$

Morphism from (X, p_X) to (Y, p_Y) is a morphism $\nu : X \to Y$ such that $p_y \circ \nu = p_X$.

This category is usually called the category of objets over Z.

[P] 6. Let X be a G-set. Consider the category $Tors_X$ which is a full subcategory of the category $GSets_X$ which consists of torsers.

Show that the category $Sh_G(X)$ is canonically equivalent to the category of functors $Funct(Tors_X \to Vect_k)$.

This is a generalization of problem 5 which is essentially equivalent to it.

[P] 7. Let G be a group and U a homogeneous G-set. Fix a point $a \in U$ and denote by H the isotropy subgroup $H = Stab(a, G)$.

(i) Show that the category $Sh_G(U)$ is canonically equivalent to the category $Rep(H)$ of representations of the group H.

(ii) More generally, suppose U is a homogeneous G-set as before but in addition we fixed a morphism of G-sets $p : X \to U$. Consider the fiber $F = p^{-1}(a)$ - it has a natural structure of an H-set.

Show that the category $Sh_G(X)$ is canonically equivalent to the category $Sh_H(F)$.

[P] 8. Suppose that in problem 7 we fixed two points $a, b \in U$. Denote by H, F the corresponding isotropy subgroups. Then the result in problem 7 (i) shows that the categories $Rep(H)$ and $Rep(F)$ are **canonically** equivalent.

Describe this equivalence directly in terms of the set $S = \{g \in G | \ a = gb\}$ which is naturally an $(H \times F)$-set.

($\Box$)**9.** Let X be a G-set. Consider the natural maps of sets $p : G \times X \to X$ and $a : G \times X \to X$ defined by $p(g, x) = x, a(g, x) = gx$.

Show that a G-equivariant sheaf $\mathcal{F}$ on X can be defined as a pair (F, α), where F is a sheaf on X and α is an isomorphism $\alpha : p^*(F) \to a^*(F)$ which satisfies the following condition:

(*) Consider the set $Z = G \times G \times X$ and two morphisms $q, b : Z \to X$, defined by $q(g, g', x) = x$ and $b(g, g', x) = gg'x$. The morphism α induces two morphisms of sheaves $\beta, \gamma : q^*(F) \to b^*(F)$. The condition on α is that these two morphisms are equal.

In this language it is easy to describe morphisms of equivariant sheaves (work out how to do this).

Such "diagramatic" or "categorical" definition of an equivariant sheaf is very useful since it works well if we want to generalize the notion of an equivariant sheaf to other situations of similar flavor.

($\Box$)**10.** There is a possibility to describe the categories of sheaves in more algebraic way.

Let X be a finite set. Denote by $S = S(X)$ the k-algebra of k-valued functions on X with usual multiplication.

(i) Show that the category $Sh(X)$ is canonically equivalent to the category of S-modules.

(ii) There exists a similar, but a little more sophisticated, description of the category $Sh(X)$ for infinite sets X. Try to give such description.

(iii) Using (i) give an "algebraic" description of the category $Sh_G(X)$.