Problem assignment 6.

Representations of Finite Groups.

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Consider a finite field $F = \mathbf{F}_q$. We assume that $charF \neq 2$. To simplify the computations you can assume that the size q of the field F is "large".

We are going to study the representations of the group $G = SL(2, F)$.

We denote by H the subgroup of diagonal matrices in G. This group is isomorphic to F^* via the isomorphism $a \in F^* \mapsto diag(a, a^{-1}) \in H$.

We denote by B the subgroup of upper triangular matrices and by N the subgroup of unipotent matrices in B. We denote by p the natural projection $p : B \to H$ with kernel N.

Let E be a quadratic extension of the field F, so that $E \simeq \mathbf{F}_{q^2}$. We have a natural embedding $E^* \to GL(2, F)$. Let us denote by T the group of elements of norm 1 in E and also the image of this group in $G = SL(2, F)$.

For every element $g \in G$ we denote by P_g its characteristic polynomial. We say that g is regular semisimple if the polynomial P_g has distinct roots; the set of such elements we denote G'.

1. Show that $\#(G) = q(q-1)(q+1) \sim q^3$.

[P] 2. Show that the group G has the following conjugacy classes

(I) Regular conjugacy classes.

(i) Split semisimple classes.

Class γ_a of an element $a \in H, a \neq \pm 1$.

Size of the class γ_a equals $q(q+1) \sim q^2$.

Classes γ_a and $\gamma_{a'}$ are distinct unless $a' = a^{\pm 1}$. Thus the number of the split semisimple classes equals $(q-3)/2 \sim q/2$.

(ii) Elliptic semisimple classes.

Class γ_d for every element $d \in T, d \neq \pm 1$.

Size of the class γ_d equals $q(q-1) \sim q^2$

Classes γ_d and $\gamma_{d'}$ are distinct unless $d' = d^{\pm 1}$. Thus the number of the elliptic semisimple classes equals $(q-1)/2 \sim q/2$.

Non-regular conjugacy classes.

(iii) Identity class γ_1 ; its size is 1

(iii') Minus identity class γ_{-1}; its size is 1.

(iv) Unipotent classes γ_u, γ'_u . These are two conjugacy classes of non-trivial elements in the unipotent subgroup N. These classes have size $(q+1)(q-1)/2 \sim q^2/2$.

(iv') Minus unipotent classes $\gamma_{-u}, \gamma_{-u'}$ .They also have size $(q+1)(q-1)/2 \sim q^2/2$.

[P] 3. For every character χ of the group H we denote by ρ_χ its extension to the group B and we consider the induced representation $\pi_\chi = Ind_B^G(\rho_\chi)$.

Check that the character $ch(\pi_\chi)$ has the following values:

$\chi(a) + \chi^{-1}(a)$ on γ_a, $\quad$ 0 on γ_d,

$q+1$ on γ_1, $\quad$ 1 on γ_u and $\gamma_{u'}$.

Compute the values on other conjugacy classes.

[P] 4. Show that the representations π_χ have the following structure

If the character χ is **regular** i.e. $\chi^2 \neq 1$, then the representation π_chi is irreducible

For the trivial character χ the representation π_1 is the direct sum of the trivial representation **1** and an irreducible representation of degree q (we will call it the Steinberg representation and denote St).

If χ is a Legendre character of H (the unique non-trivial character such that $\chi^2 = 1$), then the representation π_χ is a sum of two non-equivalent irreducible representations π^+, π^- of dimension $(q+1)/2$.

Write formulas for the characters of these representations.

[P] 5. Consider representations $\Pi_\chi = Ind_H^G(\chi)$.

Compute the character table of these representations.

Show that the character $R_\chi = ch(\Pi_\chi) - ch(\pi_\chi)$ is 0 on all regular elements.

Show that the character R_χ essentially does not depend on χ.

Namely show that its values on all unipotent classes do not depend on χ and values on minus unipotent classes are obtained from values on unipotent classes by multiplication by $\chi(-1)$.

Definition. Consider characters θ of the group T. We call such character **regular** if $\theta^2 \neq 1$. We call characters θ', θ **conjugate** if $\theta' = \theta^{\pm 1}$.

6. For any character θ of the group T consider the representation $\Pi_\theta = Ind_T^G(\theta)$. Compute the character table of these representations.

For any character θ of the group T consider the character $\pi_\theta = ch\Pi_\theta - R$ where we take $R = R_\chi$ with character χ chosen to match θ on -1, i.e. $\chi(-1) = \theta(-1)$.

Here we consider π_θ as a function on G. By definition it lies in the lattice $\mathcal{C}h(G)$ of characters of the group G.

7. Compute the character table of characters π_θ and see that $\pi_{\theta'}$ equals π_θ iff θ' is conjugate to θ.

8. Show that the scalar product of characters $< \pi_{\theta'}, \pi_\theta >$ equals

0 if θ' and θ are not conjugate

1 if $\theta' = \theta$ is regular

2 if $\theta' = \theta$ is not regular

[P] 9. Show that the characters π_θ have the following structure

(i)If θ is regular then π_θ is the character of an irreducible representation of degree $q-1$.

(ii) If $\theta = 1$ then $\pi_\theta = St - 1$

(iii) If θ is the Legendre character of the group T (i.e. the unique non trivial character such that $\theta^2 = 1$) then π_θ is the sum of characters of two non-equivalent irreducible representations π_T^+, π_T^- of dimension $(q-1)/2$.

Compute the characters of these representations.

[P] 10. Describe completely the set $Irr(G)$ and write the corresponding character table.