Problem assignment 7.

Representations of Finite Groups.

Joseph Bernstein May 3, 2007.

Consider a finite field $F = \mathbf{F}_q$. To simplify the computations you can assume that the size q of the field F is "large".

We are going to study the representations of the group $G = GL(2, F)$. The method is the same as for the group $SL(2, F)$, and we will see that the classification or irreducible representations for the group G is a little simpler.

We denote by Z the center of G; it consists of scalar matrices.

We denote by G' the subset of regular elements of G (elements with distinct eigenvalues).

1. Show that $\#G = q(q+1)(q-1)^2, \#Z = q-1$.

We denote by H the subgroup of diagonal matrices in G.

2. (i) Show that there exists an element $g \in G$ which normalizes H such that Adg defines a non-trivial automorphism w of H.

We call elements $a, a' \in H$ conjugate if $a' = a$ or $a' = w(a)$. We call an element $a \in H$ singular if $w(a) = a$.

(ii) Show that a is singular iff it lies in the subgroup Z and regular iff it lies in $H' = H \bigcap G'$.

We call characters χ, χ' of the group H conjugate if $\chi' = \chi$ or $\chi' = w(\chi)$. We call a character χ singular if $w(\chi) = \chi$.

(iii) Show that the character χ of H is singular iff it is a character of the determinant (and hence extends to the group G).

3. Let E be a quadratic extension of F. Show how to realize the group E^* as a subgroup of G.

We will denote this group T. The characters of T we usually denote by θ.

Show that the statements (i), (ii), (iii) of problem 2 hold also for the subgroup $T \subset G$.

4. Show that the conjugacy classes of G are as follows:

I. Regular elements

$\{\gamma_a | a \in H'\}$, a is defined up to conjugation.

$\{\gamma_d | d \in T'\}$, d is defined up to conjugation

II. Singular elements

$\{\gamma_z | z \in Z\}$, $\{\gamma_{zu} | z \in Z\}$, where u is the standard unipotent matrix.

[P] 5. Let χ be a character of H.

(i) Define the representation of principal series π_χ and compute its character table.

(ii) Compute scalar products of representations of principal series and describe their decomposition into irreducible components.

(iii) Define the representation $\Pi_\chi = Ind_H^G(\chi)$ and compute its character table.

Consider the character function $R_\chi = \pi_\chi - \Pi_\chi$ and check that it depends only on the restriction of the character χ to the subgroup Z.

[P] 6. Let θ be a character of the group T.

(i) Define the representation Π_θ and compute its character table,

(ii) Write down the character function π_θ.

Compute scalar products of these functions. Using these scalar products describe the decomposition of these functions in terms of characters of irreducible representations.

(iii) Finish the classification of irreducible representations of the group G.

[P] 7. (i) Let $p : X \to Y$ be a map of finite sets. Prove the following **projection formula**.

Let $\mathcal{F}$ be a sheaf on X and $\mathcal{H}$ a sheaf on Y. Then the sheaves $p_*(\mathcal{F} \otimes p^*(\mathcal{H}))$ and $p_*(\mathcal{F}) \otimes \mathcal{H}$ are canonically isomorphic.

(ii) Show the same result for equivariant sheaves. Deduce from it the projection formula for induced representations

$\pi \otimes Ind_H^G(\tau) = Ind_H^G(Res_G^H(\pi) \otimes \tau)$

Let G be a finite group and (π, G, V) its representation. We say that π is a **p-representation** (short for permutation representation) if there exists a subset $B \subset V$ which is G-invariant and is a basis of V. In other words, this means that π is isomorphic to a representation π_X for some G-set X.

We denote by $R_p(G)$ the subgroup of the representation ring $R(G)$ spanned by p-representations.

[P] 8. (i) Show that restriction and induction functors map p-representations into p-representations.

(ii) Show that for any element $b \in R_p(G)$ the element $\#G \cdot b$ is an integer linear combination of characters induced from trivial representations of cyclic subgroups.

(iii) Show that the rank of the group $R_p(G)$ equals to the number of conjugacy classes of cyclic subgroups of G.

[P] 9. Denote by $\mathbf{R_Q}(G)$ the subgroup of $R(G) = \mathcal{C}h(G)$ which consists of characters that take integer values on all elements of G.

(i) Show that a function $f \in \mathcal{C}h(G)$ which takes only rational values lies in $R_{\mathbf{Q}}(G)$.

(ii) Show that the results (i), (ii), (iii) of problem 8 hold if we replace $R_p(G)$ by $R_{\mathbf{Q}}(G)$.

Remark. In order to prove this you will need the following result from Galois theory:

Lemma. Let ξ, ξ' be two primitive n-th roots of 1. Then there exists an automorphism of the field $L = \mathbf{Q}(\xi)$ which maps ξ to ξ'.

This lemma implies that for a cyclic group H any function $f \in R_{\mathbf{Q}}(H)$ has the same value on all generators of H.

(iii) Show that the subgroup $R_p(G)$ has finite index in $R_{\mathbf{Q}}(G)$. Give some effective bound for this index.