Problem assignment 9.

Representations of Finite Groups.

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I. Variations on Jordan-Hoelder theory.

Fix an algabra A with 1 and consider the category $\mathcal{M} = \mathcal{M}(A)$ of (left, unital) A-modules.

We denote by $Irr(\mathcal{M})$ the set of isomorphism classes of simple objects in $\mathcal{M}$.

Definition. For every A-module M we define the Jordan-Hoelder content of M to be the subset $JH(M) \subset Irr(\mathcal{M})$ which consists of all simple modules which are isomorphic to subquotients of the module M.

($\square$)**1.** Show the following properties of Jordan-Hoelder content

(i) If $L \subset M$ a submodule, then $JH(M) = JH(L) \bigcup JH(M/L)$.

(ii) Let $M_\alpha \subset M$ be a system of submodules such that $M = \bigcup_\alpha M_\alpha$. Then $JH(M) = \bigcup JH(M_\alpha$.

(iii) $JH(M) = \emptyset$ iff $M = 0$.

Hint. In the proof one needs the following lemma which easily follows from Zorn's lemma

Lemma. Let M be a non-zero finitely generated A-module. Then it has a simple quotient.

Remark. In case of a module M of finite length we can introduce a more precise invariant $JH'(M)$ which is a multiset in $Irr(\mathcal{M})$, i.e. it gives a multiplicity to every element of $\rho \in Irr(\mathcal{M})$.

Probably one can develop similar notion for arbitrary module M.

II. Separation by simple modules.

[P] 2. Let A be a finite-dimensional algebra over an algebraically closed field k. Fix an element $a \in A$.

(i) Suppose we know that the two-sided ideal $J = AaA$ contains an element b which is not nilpotent. Show that there exists a simple A-module M such that $aM \neq 0$.

(ii) Conversely, prove that if for some simple A-module M we have $aM \neq 0$, then the left ideal $Aa \subset A$ contains an element b which is not nilpotent.

($\square$)(iii) Show the same statements (i), (ii) for the case when $k = \mathbf{C}$ and the algebra A has countable dimension.

In the proof you will need the following:

Lemma. Let V be a complex vector space of countable dimension and $T \in End(E)$. Suppose the operator T is not locally nilpotent. Then there exists a complex number c such that the operator $T - c$ is not invertible.

III. Reduction to subalgebra.

Let A be an algebra with 1. Fix an idempotent $e \in A$ (i.e. $e^2 = e$). We denote by $Irr(A)_e$ the subset of $Irr(A)$ which consists of modules L such that $eL \neq 0$.

Our goal is to show that the study of these modules can be reduced to study of modules over a smaller algebra B.

Namely consider the algebra $B = eAe$. Note that B does not contain the unit element $1 \in A$, but as an abstract algebra it is an algebra with an identity element (namely element e).

We define the restriction functor $R : \mathcal{M}(A) \to \mathcal{M}(B)$ by $R(M) = eM$. Equivalent description $R(M) = \{m \in M | em = m\}$).

We can give a different description of the algebra B and the functor R.

3'. Consider A-module $T = Ae$.

(i) Show that the algebra of endomorphisms $End_A(T)$ is canonically isomorphic to the opposite algebra B^0.

(ii) Show that the functor $R : \mathcal{M}(A) \to \mathcal{M}(B)$ above can be described as follows:

$R(M) = Hom_A(T, M)$ with the natural action of the algebra B induced by the right action of B on T.

(iii) Show that the functor R is exact.

[P] 3. (i) Show that if M is a simple A-module then $R(M)$ is either 0 or is a simple B-module.

(ii) Show that the corresponding map of sets $R : Irr(A)_e \to Irr(B)$ is an imbedding.

(iii) Show that the map of sets $R : Irr(A)_e \to Irr(B)$ is epimorphic.

Hint. First prove the following

Lemma. Let M be an A-module and $N = R(M)$. Then for any B-submodule $S \subset N$ there exists an A-submodule $L \subset M$ such that $R(L) = S$.

IV. Partitions.

Consider the set $I = (1, 2, ..., n)$. Decomposition x of I is the presentation of I as a disjoint union of non-empty subsets $I_1, ..., I_r$. The sizes of these sets define a partition $\lambda = \lambda(x)$ of the number n; we say that x has type λ.

Definition. Two partitions x, y of the set I we call **transversal** if together they separate points of the set I.

Two partitions λ, μ of the number n we call **transversal** if there exists a pair of transversal decompositions x, y of types λ and μ.

[P] 4. (i) Show that partitions λ, μ are transversal iff $\lambda \leq \mu^t$ (μ^t is the dual partition).

In other words the set of all partitions transversal to λ is the segment $J_\lambda = \{\mu | \mu \leq \mu^t\}$.

(ii) Show that if $\lambda = \mu^t$, then the symmetric group S_n acts simply transitively on the set of pairs of transversal partitions (x, y) of types λ, μ.

V. Gelfand pairs.

[P] 5. Let H be a finite group and $B \subset H$ its subgroup. Show that (H, B) is a strong Gelfand pair iff the pair of groups $G = H \times B$, $D = \Delta(B) \subset G$ is a Gelfand pair.

In fact prove that the corresponding algebra $\mathcal{H}(G//D)$ (the algebra of D-bi-invariant measures on G) is naturally isomorphic to the algebra $\mathcal{H}(H)_B$ (the centralizer of the algebra $\mathcal{H}(B)$ in $\mathcal{H}(H)$).

6. Using Gelfand's trick show that the algebra $\mathcal{H}(S_n)_{S_{n-1}}$ (see problem 5) is commutative. Use this fact to show that the restriction of an irreducible representation of the group S_n to the subgroup S_{n-1} is always multiplicity free.

[P] 7. Consider the natural sequence of imbeddings $S_1 \subset S_2 \subset S_3 \subset ... \subset S_n$ (from left to write) and use them to realize for every i the algebra $\mathcal{H}(S_i)$ as a subalgebra of the algebra $\mathcal{H} = \mathcal{H}(S_n)$.

Let us denote by U the subalgebra of $\mathcal{H}$ generated by centers of all algebras $\mathcal{H}(S_i)$.

Show that the subalgebra U is commutative. Show that it is a maximal commutative subalgebra, i.e. it coincides with its centralizer in $\mathcal{H}$.