Problem assignment 1.

Algebraic Theory of D-modules.

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A remark on different kinds of problems. In all my home assignments I will use the following system.

The problems without marking are just exercises. You have to convince yourself that you can do them but it is not necessary to write them down (if you have difficulties with one of these problems ask me or Dmitry).

The problems marked by [**P**] you should hand in for grading.

The sign ($*$) marks more difficult problems.

The sign ($\square$)marks more challenging and more interesting problems which are related to some interesting subjects. They are not always related to the course material, but I definitely advise you to think about these problems.

1. (i) Prove that the algebra D_n of differential operators on R^n with polynomial coefficients is generated by $x_1, ..., x_n, \partial_1, ..., \partial_n$ with relations $[x_i, x_j] = 0$, $[\partial_i, \partial_j] = 0$, $[\partial_i, x_j] = \delta_{ij}$.

(ii) Prove that the monomials $x^\alpha \partial^\beta$ form a basis of D_n.

2. Here is the definition of topology on $C_c^\infty(\mathbf{R})$ in terms of sequences:
$\phi_i \to \phi$ iff

(i) There exists a compact subset $K \subset X$ such that for any i, $supp(\phi_i) \subset K$

(ii) For any differential operator d the sequence of functions $d\phi_i$ converges to the function $d\phi$ absolutely and uniformly.

The task is to define this topology in terms of open sets.

3. Show that the completion of the space $C_c^\infty(\mathbf{R}^n)$ in weak $C_c^\infty(\mathbf{R}^n)$ topology is isomorphic to the dual space $(C_c^\infty(\mathbf{R}^n))^*$.

4. i) Define multiplication of a generalized function by a smooth function.

ii) Define partial derivatives of generalized functions.

iii) For any differential operator d with smooth coefficients describe explicitly the action of d on generalized functions in terms of distributions.

5. Let

$$Q(x) := \sum_{i=1}^{k} x_i^2 - \sum_{i=k+1}^{n} x_i^2.$$

Let Θ be a connected component of the set $\mathbf{R}^n \setminus Zeroes(Q)$. Show that the distribution Q_Θ^λ has meromorphic continuation to the entire complex plane.

Two proofs were outlined in the class.