Problem assignment 3.

Algebraic Theory of D-modules.

Joseph Bernstein

1. Exterior tensor product of D-modules.

Consider two affine spaces X, Y. Let M be a $D(X)$-module and N be a $D(Y)$- module.

(i) Consider the vector space $M \otimes_K N$ and define on it the structure of a $D(X \times Y)$-module. This module is called the **exterior tensor product** of M and N, the standard notation is $M \boxtimes N$.

(ii) Show that $d(M \boxtimes N) = d(M) + d(N)$ and $e(M \boxtimes N) = e(M)e(N)$.

2. Tensor product over $\mathcal{O}$. Let M, N be two $D(X)$-modules. Consider the space $M \otimes N := M \otimes_{\mathcal{O}(X)} N$ and define on it the structure of a $D(X)$-module using the Leibnitz rule. This defines on D-modules the bifunctor of (inner) **tensor product**.

(i) Show that $M \otimes N$ is canonically isomorphic to the module $\Delta^0(M \boxtimes N)$ where Δ is the diagonal imbedding $\Delta : X \to X \times X$.

(ii) Interpret the analytic meaning of this algebraic operation. Convince yourselves that the inner tensor product of finitely generated D-modules is not always finitely generated.

[P] (iii) Show that if M, N are holonomic then the tensor product $M \otimes N$ is also holonomic.

[P] 3. Tensor product of left and right D-modules. Let M be a left $D(X)$-module and R a right $D(X)$-module.

Consider the tensor product $M \otimes_{\mathcal{O}(X)} R$ and define on it the natural structure of a **right** $D(X)$-module.

Explain the analytic meaning of this construction. Convince yourselves that there is no natural tensor product of right D-modules.

Show that the tensor product of holonomic modules is holonomic.

[P] 4. Let X be a real vector space X. Fix a real polynomial P on X and consider as before the family of functions P^λ.

Let f be a generalized function on X. We assume that the function f is not very singular (e.g. it is a Schwartz function, or more generally, it is a finite sum of differential operators applied to continuous functions). In this case we can define the family of generalized functions $G(\lambda) := P^\lambda f$ for $\mathbf{Re}\lambda >> 0$.

Show that if the function f is holonomic then the family $G(\lambda)$ extends as a meromorphic function of λ to the whole complex plane.

More precisely, show that these family of functions satisfies a recurrence relation of the following form

(*) There exist a differential operator $d \in D(X)[\lambda]$ and a polynomial $b \in C[\lambda]$ such that the following identity holds $d(\lambda)G(\lambda + 1) = b(\lambda)G(\lambda)$.

[P] 5. (i) Similarly to problem 4 show that the function $G(\lambda, \mu) := P^\lambda Q^\mu f$ has meromorphic continuation in two variables λ, μ.

($\Box$)(ii) Show that in this case the function $b(\lambda, \mu)$ can be chosen as a product of linear functions.

[P] 6. Let T be a differential operator with constant coefficients on $X = \mathbf{R}^n$. We would like to find a fundamental solution F for T, i.e a distribution solution of the differential equation $T(F) = \delta$ (delta function).

Using results discussed in class show that such solution always exists.

Show that we can choose F to be a Schwartz distribution.

Show that we can choose F to be a holonomic distribution.

[P] 7. Let P be a real polynomial on $X = \mathbf{R}^n$. Consider on X a function $H(x) = \exp(iP(x))$ and denote by F its Fourier transform.

Show that F is a holonomic distribution. Later we will see that this implies that F is analytic outside some explicitly described analytic subset.

[P] 8. Let P be a real polynomial on $X = \mathbf{R}^n$. Let us assume that it is strictly positive and grows at infinity (for example assume that $P \geq (1+\sum x_i^2)$).

(i) Consider the function $F(\lambda) := \int_X P^\lambda dx$. This function is defined for $\mathbf{Re}\lambda << 0$.

Show that the function F expends to a meromorphic function on the whole complex plane. To do this show that the function F satisfies a recurrence relation of the following shape

(**) There exists a number m and rational functions $a_i(\lambda)$ for $i = 0, 1, ..., m$ such that the following identity holds

$$F(\lambda + m + 1) = \sum_{i=0}^{m} a_i(\lambda) F(\lambda + i)$$

(ii) Fix a holonomic Schwartz distribution E and consider the function $F = F_E$ defined by $F(\lambda) := \int_X P^\lambda E$.

Prove the same results for this function.