Problem assignment 4.

Algebraic Theory of D-modules.

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In this assignment we fix an algebra A and let $\mathcal{M}(A)$ denote the category of left A-modules.

Definition. (i) Let M be an A-module. We say that M is **finitely generated** if it has a finite system of generators as an A-module. We say that M is **Noetherian** if any submodule of M is finitely generated.

(ii) We say that the algebra A is (left) Noetherian if the category $\mathcal{M}(A)$ is Noetherian, which means that every finitely generated object in this category is Noetherian.

1. (i) Show that an A-module M is finitely generated iff it satisfies

(*) For any system of submodules $M_\kappa \subset M$ which is exhaustive (i.e. $\sum M_\kappa = M$) there exists a finite exhaustive subsystem.

(ii) Show that any A-module M can be obtained as a directed direct limit of finitely generated submodules (see definition below).

Show that the algebra A is left Noetherian iff it is Noetherian as a left A-module.

Definition. Let I be a partially ordered set. We will consider it as a category with one morphism $i \to j$ for every pair of objects $i, j \in I$ such that $i \leq j$.

We define an I-system F of objects in $\mathcal{M}$ as a functor $F : I \to \mathcal{M}$

2. (i) Define the notion of a direct limit $\lim_d(F)$ for an I-system F.

(ii) Show that if the system I is **directed** then the functor $F \mapsto \lim_d(F)$ is exact (the system I is called directed if for any pair of elements $i, j \in I$ there exists an element k such that $i \leq k, j \leq k$.)

3. Show that an object $T \in \mathcal{M}(A)$ is finitely generated iff the functor $H_T : \mathcal{M}(A) \to Ab$ given by $H_T(N) = Hom(T, N)$ commutes with arbitrary directed direct limits. Show that in this case it commutes with any direct limits.

4. (i) Fix some map $D : A \to A$ and consider the algebra B generated by A and an element t subject to relations $ta - at = D(a)$ for $a \in A$.

Show that if the algebra A is Noetherian then the algebra B is also Noetherian.

(ii) More generally, fix an automorphism $Q : A \to A$ and a map $D : A \to A$ and define the algebra B via relations $ta = Q(a)t + D(a)$.

Prove that the same conclusion holds.

5. Consider an Abelian category $\mathcal{M}$ (for example a category $\mathcal{M}(A)$ of A-modules for some algebra A). Let K, L be two objects in $\mathcal{M}$. Suppose $P^\cdot$ is a projective resolution of K and $Q^\cdot$ is a projective resolution of L. Consider $P^\cdot, Q^\cdot$ as objects in the homotopy category $Homt(\mathcal{M})$.

Consider the canonical homomorphism $\alpha : Hom_{Homt}(P^\cdot, Q^\cdot) -> Hom_{\mathcal{M}}(k, L)$.

Show that α is an isomorphism.

In fact show that the same is true even if we do not assume that the resolution $Q^\cdot$ is projective.

6. Consider a short exact sequence $0 -> N' -> N -> N'' -> 0$ in $\mathcal{M}$. Fix an object $K \in \mathcal{M}$.

(i) Prove the long exact sequence
$0 \to Ext^0(K, N') \to Ext^0(K, N) \to Ext^0(K, N'') \to$
$\to Ext^1(K, N') \to Ext^1(K, N) \to Ext^1(K, N'') \to$
$\to Ext^2(K, N') \to Ext^2(K, N) \to ...$

(ii) Prove the long exact sequence
$0 \to Hom(N'', K) \to Hom(N, K) \to Hom(N', K) \to$
$\to Ext^1(N'', K) \to Ext^1(N, K) \to Ext^1(N', K) \to$
$\to Ext^2(N'', K) \to Ext^2(N, K) \to ...$

7. Let A be a Noetherian algebra and M a Noetherian A-module. Show that the functor $N \mapsto Ext^i(M, N)$ commutes with directed direct limits.

How do you think, does it commute with arbitrary direct limits ?

8. Let $\mathfrak{g}$ be a finite dimensional Lie algebra and $U = U(\mathfrak{g})$ its universal enveloping algebra. Show that U has finite cohomological dimension $\leq \dim \mathfrak{g}$.