**Problem assignment 5.**

**Algebraic Theory of $D$-modules.**

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In this assignment we fix an algebra $A$ and let $\mathcal{M}(A)$ denote the category of left $A$-modules.

**1.** Suppose that the algebra $A$ is Noetherian. We denote by $\mathcal{M}^f(A)$ the category of finitely generated $A$-modules.

Show that an $A$-module $P \in \mathcal{M}^f(A)$ is projective in category $\mathcal{M}(A)$ iff it is projective in the category $\mathcal{M}^f(A)$.

More generally, show that if cohomological dimension of $P$ is bounded by $d$ in small category $\mathcal{M}^f(A)$, then it is bounded by $d$ in the large category $\mathcal{M}(A)$.

**[P] 2.** Let $A$ be a Noetherian algebra and $M$ a finitely generated $A$-module. Suppose we know that $chdim(M) \leq d$.

Consider the functor $E : N \mapsto Ext^d(M, N)$. Show that there exists a right $A$-module $R$ such that this functor is isomorphic to the functor $T_R$ defined by $T_R(N) = R \otimes_A N$. Show that the module $R$ is defined uniquely up to canonical isomorphism.

**Definition**. Suppose we are given a vector space $V$ over a field $K$.

Let us define a **finiteness structure** $F$ on $V$ to be a finite collection of commuting operators $x_i : V \to V$, $i = 1, ..., n$ which define on $V$ a structure of a **finitely generated** module over the algebra $A = K[x_1, ..., x_n]$.

Two finiteness structures $F = (x_i)$ and $H = (y_j)$ we call **equivalent** if operators $x_i$ and $y_j$ commute.

**[P] 3.** (i) Show that a finiteness structure on $V$ allows unambiguously define the functional dimension $d(V)$.

**[P] 4.** Let $F$ be a finiteness structure on $V$; it is given by a structure of an $A$-module on $V$ where $A = K[x_1, ..., x_n]$.

For a given integer $l$ consider the vector space $D^l(V) = D^l_F(V)$ defined by $D^l(V) := Ext_A^{n-l}(V, \omega_A)$, where $\omega_A$ is the $A$-module $\Omega^n(A)$ of highest degree differential forms on the affine space $X$ corresponding to the algebra $A$. By construction this is a space with a finiteness structure.

(i) Show that for two equivalent finiteness structures $F$ and $H$ on $V$ the spaces $D^l(V)$ constructed using $F$ and using $H$ are **canonically isomorphic**.

(ii) Show that $d(D^l(V) \leq l$

(iii) Show that if $d(V) = d$ then $D^l(V) = 0$ for $l > d$.

(iv) Show that if $d(V) = 0$, i.e. $V$ is a finite dimensional vector space over $K$, then the space $D^0(V)$ is just the dual vector space $V^*$.

($\square$)**5.** Is this construction of the space $D^l(V)$ compatible with the composition of equivalences ?

**[P] 6.** Consider a commutative algebra $A$ and an $A$-module $M$.

Let $B$ be a finitely generated subalgebra of $A$ (or, more generally, a finitely generated commutative algebra together with a morphism $\nu : B \to A$).

We say that the module $M$ is $B$-finite if it is finitely generated as $B$-module.

We say that $M$ is CM (Cohen-Macaulay) of dimension $d$ if for some polynomial subalgebra $B$ in $d$ variables $M$ is $B$-finite and projective.

Show that in this case for any polynomial algebra $B$ in $d$ variables if $M$ is $B$-finite then it is automatically $B$-projective.

**7.** let $\mathfrak{g}$ be a finite-dimensional Lie algebra over the field $K$. Show that the category $\mathcal{M}(\mathfrak{g})$ of $\mathfrak{g}$-modules is Noetherian and has finite cohomological dimension.