Problem assignment 6.

Algebraic Theory of D-modules.

Joseph Bernstein

In this assignment we fix an algebra A and let $\mathcal{M}(A)$ denote the category of left A-modules.

[P] 1. Consider the case of the algebra $A = K[x_1, ..., x_n]$, where K is a field (for simplicity we assume K to be infinite).

(i) Show that any finitely generated A-module M is glued from modules of the type $M = A/J$.

(ii) Show that if $M = A/J$ and the ideal J is not zero, then after an appropriate change of variables we can assume that M is finitely generated over the subalgebra $B = K[x_1, ..., x_{n-1}] \subset A$.

Prove that in this case A-module $E^i_A(M) := Ext^i_A(M, A)$ is finitely generated over B and as a B-module it is isomorphic to $E^{i-1}_B(M)$.

(iii) Using this fact prove that $d(E^i_A(M)) \leq n - i$.

Also prove that for $i < n - d(M)$ the module $E^i_A(M)$ is zero.

(*) **2.** Let A be a finitely generated commutative algebra over an algebraically closed field K.

Set $X = Specm(A)$. For every point $x \in X$ we consider the maximal ideal $\mathfrak{m}_x$ corresponding to x, the A-module $K_x = A/\mathfrak{m}_x$ and the Zariski cotangent space $T^*_x X = \mathfrak{m}_x/\mathfrak{m}^2_x$.

Show that the following conditions are equivalent:

(i) X, A is non-singular near the point x.

(ii) A-module K_x has finite cohomological dimension.

(iii) In some neighborhood of the point x there exists a coordinate system, i.e. a system of regular functions $x_1, ..., x_n \in \mathfrak{m}_x$ and vector fields $\partial_1, ..., \partial_n$ such that $\partial_i(x_j) = \delta_{ij}$ and differentials dx_i span the Zariski cotangent space $T^*_x X$.

[P] 3. Let A be a filtered algebra such that the associated graded algebra Σ is commutative and Noetherian.

Fix a finitely generated A-module M, choose its good filtration and denote by M_Σ the associated graded module. Consider ideals $J = Ann(M_\Sigma) \subset \Sigma$ and $R = Rad(J) \subset \Sigma$.

Show that an element $f \in \Sigma^l$ belongs to R iff it satisfies the following condition

(*) For any element $m \in M$ $deg(f^k m) - kl \to -\infty$ when $k \to \infty$.

($\Box$)**4. Commutative subalgebras of a Weyl algebra.**

Consider the Weyl algebra $A = A(n)$ with the arithmetic filtration and denote by Σ its associated graded algebra. This is the standard graded polynomial algebra in $2n$ variables. We set $X = Specm(\Sigma) \approx \mathbb{A}^{2n}$

(i) Describe the symplectic structure on the space X corresponding to the Poisson bracket on the algebra Σ.

(ii) Let $C \subset A$ be a commutative subalgebra. Consider the corresponding subalgebra $C_\Sigma \subset \Sigma$.

Show that this subalgebra is Poisson commutative.

(iii) In (ii) deduce that the algebra C_Σ has dimension $\leq n$ (this means that any finitely generated subalgebra of it has functional dimension $\leq n$).

Deduce from this that any commutative subalgebra $C \subset A(n)$ has dimension $\leq n$.

Can you think of other proofs of this fact.

(□)**5. Non-holonomic irreducible D-modules.**

In this problem we work with the Weyl algebra $A = A(2)$, so the space X is 4-dimensional

(i) Fix a homogeneous polynomial $P \in \Sigma$ of degree k. Let $Z \subset X$ be the homogeneous variety of its zeroes and $P(Z) \subset P(X) = \mathbb{P}^3$ its projectivization.

Consider the polynomial P as a Hamiltonian and denote by ξ the corresponding vector field on X. Check that this field preserves the subvariety Z and defines a direction field D on the surface $P(Z)$.

We say that the polynomial P is non-degenerate if the surface $P(Z)$ is non-singular and the direction field D on the algebraic surface $P(Z)$ does not have algebraic solutions (i.e. there is no algebraic curve $C \subset P(Z)$ which is tangent to D).

One can show that generic polynomial P of degree $k \geq 3$ has this property.

(ii) Suppose P is a non-degenerate homogeneous polynomial of degree k. Choose any element $b \in A^k$ such that its symbol $\sigma^k(b)$ equals P.

Show that in this case the left ideal $J = Ab$ is a maximal left ideal of the algebra A. In particular, show that the A-module $M = A/J$ is simple and has functional dimension $d(M) = 3$, i.e. it is not holonomic.