Problem assignment 7.

Algebraic Theory of D-modules.

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Digression on Algebraic Geometry.

In this assignment we fix an algebraically closed field k. We will discuss properties of affine algebraic varieties over k.

Definition. (i) An affine algebraic variety is a pair (X, A) where X is a set and $A = \mathcal{O}(X)$ a k-subalgebra of the algebra $k[X]$ of all k-valued functions on X which satisfy

(a) A is a finitely generated k-algebra

(b) The map $x \mapsto \nu_x : A \to k$ is a bijection of X with the set of homomorphisms of k-algebras $A \to k$.

A morphism $\pi : X \to Y$ of affine algebraic varieties is a map of sets $\pi : X \to Y$ such that the corresponding morphism π^* on functions maps $\mathcal{O}(Y)$ into $\mathcal{O}(X)$

For any ideal $J \subset A = \mathcal{O}(X)$ we denote by $Z(J) \subset X$ the set of its zeroes. Similarly for any subset $Z \subset X$ we denote by $J(Y) \subset A$ the ideal of function that vanish on Z.

We define the **Zariski** topology on X by condition that a subset $Z \subset X$ is closed if $Z(J(Z)) = Z$.

Definition. A morphism $\pi : X \to Y$ of affine algebraic varieties is called **finite** if $\mathcal{O}(X)$ is a finitely generated $\mathcal{O}(Y)$ module.

1. Let $\pi : X \to Y$ be a finite morphism. Show that its fibers are finite and its image is closed. Namely show that $Im(\pi) = Z(J)$, where $J = Ker(\pi^*)$.

In particular, show that a finite morphism π is epimorphic iff $\pi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ is injective.

Definition. (i) Let X, A be an affine algebraic variety and M a finitely generated A-module. We define the **support** of module M to be $Supp(M) := Z(J_M) \subset X$, where $J(M)$ is the annihilator of M in A.

(ii) We define $d(M)$ to be the functional dimension of the module M.

(iii) For any closed subset $Z \subset X$ we define its **Hilbert dimension** to be $\dim_H(Z) := d(\mathcal{O}(Z))$.

2. Show that $d(M) = \dim_H(Supp(M))$.

The following problem provides an axiomatic definition of the notion of dimension for affine algebraic varieties.

3. (i) Show that there exists no more than one function $X \mapsto \dim X$ from affine algebraic varieties to integers that satisfies

(a) If $\pi : X \to Y$ is a finite morphism then $\dim X \le \dim Y$. If in addition it is epimorphic then $\dim X = \dim Y$.

(b) $\dim \mathbf{A}^d = d$, $\dim(\emptyset) = -\infty$

(ii) Show that the function $\dim X = \dim_H(X)$ has these properties.

Definition. (i) An affine algebraic variety X is called **irreducible** if it can not be presented as a union of two closed subsets strictly smaller than X.

(ii) An irreducible chain of length l in X is a collection of closed irreducible subsets $Z_0 \subsetneqq Z_1 \subsetneqq ... \subsetneqq Z_l \subset X$.

We define the **Krull dimension** $\dim_K X$ of X to be the maximal length of a chain of irreducible subsets in X.

4. (i) Show that any affine algebraic variety X can be canonically written as a union of its irreducible components.

(ii) Show that Krull dimension satisfies the axiomatic definition of dimension.

Definition. Let $X \subset \mathbf{A}^n$ be an affine algebraic variety. Let us define the **intersection dimension** $\dim_{int} X$ of the variety X to be the minimal number l such that the generic affine subspace $L \subset \mathbf{A}^n$ of codimension $l+1$ does not intersect X.

5. Show that the intersection dimension satisfies the axiomatic properties of dimension.

6. Consider the affine space $X = \mathbf{A}^n$, so that the algebra $A = \mathcal{O}(X)$ is a polynomial algebra. Let M be a finitely generated $\mathcal{O}(X)$-module.

(i) Show that the minimal number l such that $Ext^l(M, A) \neq 0$ coincides with the codimension of the $Supp(M)$ in X.

(ii) Prove the following **Principle ideal theorem.**

Let X be an irreducible affine algebraic variety of dimension m and $f \in \mathcal{O}(X)$ a non-zero function. Then every irreducible component of the affine algebraic variety $Z(f)$ (zeroes of the function f) has dimension equal to $m-1$.

(iii) Let $Z \subset X = \mathbf{A}^n$ be a closed algebraic subvariety of codimension r. Show that one can find r functions $f_1, ..., f_r \in J(Z)$ such that the subset $F = Z(f_1, ..., f_r)$ of their common zeroes has codimension r and contains Z.

Definition. Let Z be a subset of an affine algebraic variety X. We say that Z is **locally closed** if it is an intersection of a Zariski closed subset and a Zariski open subset.

We say that Z is **constructible** if it is a finite union of locally closed subsets

7. Prove the following **Chevalley Theorem.**

Let $\pi : X \to Y$ be a morphism of affine algebraic varieties. Then the image of X in Y is a constructible subset.

8. Using devissage and Noether normalization lemma prove a stronger version of this theorem. Namely we say that the morphism $\pi : X \to Y$ is special if for some natural number d we can decompose it into a finite epimorphism $X \to Z = \mathbf{A}^d \times Y$ and the natural projection $p : Z \to Y$.

Show that given any morphism $\nu : X \to Y$ of affine algebraic varieties we can decompose varieties X and Y into finite number of disjoint affine subvarieties $X_i \subset X$, $Y_j \subset Y$ such that for any i we can find j such that $\nu : X_i \to Y_j$ is a special morphism.