Problem assignment 8.

Algebraic Theory of D-modules.

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Definition. Let $\mathcal{A}$ be an abelian category. A full subcategory $S \subset \mathcal{A}$ is called a **Serre** subcategory if it satisfies the following condition

(*) For any short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ the object B lies in S iff A and C lie in S.

1. (i) Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor between abelian categories. Show that $S = Ker(F) := \{X \in \mathcal{A} | \ F(X) \simeq 0\}$ is a Serre subcategory.

(ii) Conversely show that any Serre subcategory $S \subset \mathcal{A}$ can be constructed as a kernel of an exact functor $F : \mathcal{A} \to \mathcal{B}$.

(iii) Show that in (ii) we can assume that the functor F is essentially onto. Show that with this condition the category $\mathcal{B}$ is defined uniquely up to canonical equivalence of categories.

This category is called the **quotient** category and usually denoted $\mathcal{B} = \mathcal{A}/S$.

2. Consider the quotient category $\mathcal{B} = \mathcal{A}/S$ described in problem 1. Given an object $T \in B$ we can consider the group $Aut(T)$ of its automorphisms.

Construct a canonical map $Aut(T) \to K(S)$ and show that this is a morphism of groups (here $K(S)$ is the K-theory of the category S).

3. Let X be an algebraic variety, $Z \subset X$ a closed subvariety and $U = X \setminus Z$ its open complement.

Consider the category $\mathcal{M}(\mathcal{O}_X)$ of $\mathcal{O}$-modules on X. Let $S = \mathcal{M}_Z(\mathcal{O}_X)$ be the subcategory of sheaves supported on Z.

Show that this is a Serre subcategory and that the quotient category is canonically equivalent to $\mathcal{M}(\mathcal{O}_U)$.

4. Let X be an algebraic variety. Consider in $\mathcal{M}(\mathcal{O}_X)$ the subcategory $S = Coh(\mathcal{O}_X)$ of coherent $\mathcal{O}$-modules.

(i) Show that S is a Serre subcategory.

(ii) Show that any $\mathcal{O}$-module is a direct limit of coherent submodules.

Hint. In order to prove (ii) use the following

Lemma. Let $\mathcal{F}$ be an $\mathcal{O}$-module on X , $U \subset X$ be an open subset. Consider $\mathcal{O}_U$-module $\mathcal{F}_U$ obtained by restriction of $\mathcal{F}$ to U.

Fix a coherent $\mathcal{O}_U$ submodule $\mathcal{C}' \subset \mathcal{F}_U$. Then it can be extended to a coherent submodule $\mathcal{C} \subset \mathcal{F}$ (so that the restriction of $\mathcal{C}$ to U equals to $\mathcal{C}'$).

Hint. First prove this when X is affine and U is a basic open subset, then for the case when X is affine X and then for general X.

5. Let V be a finite dimensional vector space over k which we consider as an algebraic variety. We denote by P the algebra of regular functions on V; we consider it as a graded algebra.

(i) Show that the grading is defined by the canonical action of the multiplicative group $H = G_m$ on the space V.

(ii) Show that the category $\mathcal{M}$ of graded finitely generated P-modules is equivalent to the category of H-equivariant coherent $\mathcal{O}$-modules on V.

(iii) Consider the subcategory $S \subset \mathcal{M}$ of $\mathcal{O}$-modules supported at 0. Show that this is a Serre subcategory and that the quotient category $\mathcal{M}/S$ is canonically equivalent to the category of coherent $\mathcal{O}$-modules on the projective space $\mathbf{P}(V)$.

This gives an "algebraic" description of this highly non-trivial category.

6. Let $\mathcal{M}$ be an abelian category and S a Serre subcategory. Define a subcategory $D_S(\mathcal{M}) \subset D(\mathcal{A})$ and show that it is a triangulated category. Construct a canonical exact functor between triangulated categories $t_S : D(S) \to D_S(\mathcal{M})$.

Let us say that S is a nice subcategory if the functor t_S is an equivalence of categories.

7. Let A be a Noetherian algebra, $\mathcal{M}$ be the category of (left) A-modules. Show that the subcategory $S \subset \mathcal{M}$ of finitely generated A-modules is nice.

Prove similar statement for the subcategory $Coh(\mathcal{O}_X) \subset \mathcal{M}(\mathcal{O}_X)$ for any algebraic variety X.

(*) **8.** For a closed subset $Z \subset X$ show that the subcategory $\mathcal{M}_Z(\mathcal{O}_X) \subset \mathcal{M}(\mathcal{O}_X)$ is nice.

Hint. Do this for the case when X is a line and Z is a point.

9. Let $\mathcal{T}$ be a triangulated category and Q a class of morphisms in T. Let us assume that Q is localizing, i.e. there exists a localized category $L = Q^{-1}\mathcal{T}$ corresponding to Q.

Let us also assume that Q is compatible with the triangulated structure. This means that

Q is invariant with respect to shift functors $[k]$

If $\nu : (X \to Y \to Z) \to (X' \to Y' \to Z')$ is a morphism of exact triangles such that morphisms ν_X, ν_Z lie in Q, then the morphism ν_Y also lies in Q.

Show that in this case the quotient category L has canonical structure of a triangulated category.

10. In problem 9 show that instead of a class of morphisms Q one can consider a subcategory $S \subset \mathcal{T}$ consisting of all cones of morphisms in Q (let us call such category a closed subcategory).

(i) Describe closed subcategories in axiomatic way and give a description of the quotient triangulated category $\mathcal{T}/S$.

(□)(ii) Let S be a Serre subcategory of an abelian category $\mathcal{M}$. Describe some conditions on S that imply that the triangulated category $D(\mathcal{M}/S)$ is equivalent to the triangulated category $D(\mathcal{M})/D_S(\mathcal{M})$.