Problem assignment 1.

Algebraic Geometry and Commutative Algebra

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A remark on problems in different areas. In my assignments I will try to single out problems that are not directly related to algebraic geometry. For example sign (CA) signifies a problem (or a definition) from commutative algebra, (LA) stands for linear algebra.

A remark on different kinds of problems. In all my home assignments I will use the following system.

The problems without marking are just exercises. You have to convince yourself that you can do them but it is not necessary to write them down (if you have difficulties with one of these problems ask me or Dmitry).

The problems marked by [**P**] you should hand in for grading.

The sign ($*$) marks more difficult problems.

The sign (∇) marks more challenging and more interesting problems which are related to some interesting subjects. They are not always related to the course material, but I definitely advise you to think about these problems.

Remark. In this assignment you can freely use the Nullstellensatz. Please specify every time you use it (for example put the sign (NS)). Check for yourself that you really need it for your argument to work.

[**P**] **1.** Let X be an affine algebraic variety and B a subset of X. Suppose we know that B is a basic open subset, i.e. it has a form $B = X_f$ for some function $f \in \mathcal{P}(X)$.

Show how to describe the subalgebra $\mathcal{P}(B) \subset \mathcal{F}(B)$ in terms of the set B, without knowing the function f that defines it.

2. Let X be an affine algebraic variety. Denote by $\mathcal{B}$ the family of basic open subsets of X.

Show that $\mathcal{B}$ forms a base of a topology and that the topology defined by $\mathcal{B}$ is the Zariski topology on X.

Definition. A topological space X is called **quasicompact** if any open covering $\{U_\alpha\}$ of X has finite subcovering.

[**P**] **3.** Show that any affine algebraic variety is quasicompact in Zariski topology.

4. (i) Let A be a finitely generated k-algebra with no nilpotents. Consider the set $X = Mor_{k-alg}(A, k)$.

Show that A embeds into the algebra $\mathcal{F}(X)$ of k-valued functions on X and that the pair (X, A) is an affine algebraic variety.

(ii) If X, Y are affine algebraic varieties, show that
$Mor(X, Y) = Mor_{k-alg}(\mathcal{P}(Y), \mathcal{P}(X))$.

Definition. (CA) Let A be a ring (commutative, with 1). Given an element $f \in A$ we define the **localization** of A with respect to f to be the ring $A_f = A[t]/(ft-1)A[t]$.

5. (CA) (i) Show that we can define the ring A_f as a set of fractions $\frac{a}{f^k}$ modulo the following equivalence relation: $\frac{a}{f^k}$ is equivalent to $\frac{b}{f^l}$ iff for some N we have $f^N f^l a = f^N f^k b$

(ii) Describe the kernel of the canonical morphism $i : A \to A_f$. In what cases the localized ring A_f is trivial, i.e. consists of one element 0 ?

(iii) Show that if A has no nilpotents (resp. no zero divisors), then A_f has no nilpotents (resp. no zero divisors).

6. Fix a finite dimensional vector space V over k and denote by $\mathbf{V}$ the corresponding algebraic variety (affine space). We would like to define a structure of an algebraic variety on the set $X = \mathbf{P}(V)$ of lines in V.

Denote by $\mathbf{V}^*$ an open algebraic subvariety $\mathbf{V}^* = \mathbf{V} \setminus 0 \subset \mathbf{V}$ and consider the natural projection of sets $p : \mathbf{V}^* \to \mathbf{P}(V)$. Define topology and a sheaf $\mathcal{O}$ on $\mathbf{P}(V)$ using this projection, and show that this is an algebraic variety.

[P] 7. (i) Describe the algebra $\mathcal{O}(\mathbf{V}^*)$ of global regular functions on $\mathbf{V}^*$.

(ii) Describe the algebra $\mathcal{O}(\mathbf{P})$ of global regular functions on $\mathbf{P}(V)$.

(iii) Let X be a variety obtained from $\mathbf{P}^2$ by removing one point. Describe the algebra $\mathcal{O}(X)$ of global regular functions on X.

8. Consider the subvariety $Y \subset \mathbf{A}^2$ defined by one equation $x^2 - y^3 = 0$.

Describe a morphism $\nu : \mathbf{A}^1 \to Y$ which is bijective. Show that it is a homeomorphism, but not an isomorphism of algebraic varieties.

9. Show that the affine space $\mathbf{A}^n$ can be realized as an open algebraic subvariety of the projective space $\mathbf{P}^n$. Describe the complementary closed subvariety $\mathbf{P}^n \setminus \mathbf{A}^n$.