Problem assignment 13.

Algebraic Geometry and Commutative Algebra

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1. Let $\pi : X \to Y$ be a morphism of algebraic varieties, F an $\mathcal{O}_X$-module. Consider the derived sheaves $R^i\pi_*(F)$ on Y.

(i) Show that if π is affine then $R^0\pi_*(F)$ is an $\mathcal{O}_Y$-module and $R^i\pi_*(F) = 0$ for $i \neq 0$.

(ii) Suppose π is separated morphism. Chose an open covering $\mathcal{U} = (U_i)$ of X consisting of open subsets affine over Y and consider the Cech complex $C_{\mathcal{U}}(F)$. Show that the complex of $\mathcal{O}_Y$ modules $\pi_*(C_{\mathcal{U}}(F))$ computes the derived sheaves $R^i\pi_*(F)$. In particular show that these sheaves are $\mathcal{O}_Y$-modules.

(iii) Show that if Y is affine then $H^i(X, F) = \Gamma(Y, R^i\pi_*(F))$.

Let V be a linear space. Consider the standard diagram of morphisms $p : V^* \to \mathbf{P}(V)$ and $j : V^* \to \mathbf{V}$.

2. Let M be an $\mathcal{O}$-module on V^*. Show that for $i > 0$ the sheaf $R^ij_*(M)$ is an $\mathcal{O}_{\mathbf{V}}$-module supported at the point 0.

Using this fact show that the action of the polynomial algebra $\mathcal{O}(\mathbf{V})$ on the space $H^i(V^*, F)$ is locally nilpotent when restricted to the maximal ideal of the point 0.

3. Serre's computation of cohomologies $H^i(V^*, \mathcal{O}_{V^*})$.

Choose coordinates $(x_1, x_2, ..., x_n)$ on $\mathbf{V}$.

(i) In case $n = 1$ consider the complex R of $\mathcal{O}_{\mathbf{V}} = k[x]$-modules

$0 \to \mathcal{O}_{\mathbf{V}} \to \mathcal{O}_{V^*} \to \Delta \to 0$ and describe explicitly the module Δ.

(ii) For an arbitrary $n > 1$ consider the complex $R^n = R \otimes R \otimes R ... \otimes R$ that we consider as a complex of $\mathcal{O}_{\mathbf{V}} = k[x_1, ..., x_n]$-modules. Show that it is exact.

Compare this complex with the Cech resolution for computation of cohomologies $S^i = H^i(V^*, \mathcal{O}_{V^*})$.

Using this show that as $\mathcal{O}_{\mathbf{V}}$-modules $S^0 = \mathcal{O}_{\mathbf{V}}$, $S^{n-1} = \Delta^{\otimes n}$ and $S^i = 0$ for other i-s.

4.. Let F be a coherent $\mathcal{O}$-module on $\mathbf{P}(V)$.

(i) Show that for large k the twisted $\mathcal{O}$-module $F(k)$ is acyclic.

(ii) Show that we can embed F into a coherent acyclic $\mathcal{O}$-module.

(iii) Show that we can find a resolution of F of the shape $0 \to Q_1 \to \mathbf{Q}_2 \to ...$ consisting of coherent acyclic $\mathcal{O}$-modules.

(iv) Show that we can choose a resolution Q above to be of length $n-1$, where $n = \dim V$.

∇ (v) Show that we can choose a resolution $Q = Q(F)$ in finitely functorial way. This means the for any finite diagram D of coherent $\mathcal{O}$-modules and their morphisms we can lift this diagram to the diagram of corresponding resolutions Q

5. Let F be a coherent $\mathcal{O}$-module on $\mathbf{P}(V)$. Show that for large k the dimension $\dim \Gamma(\mathbf{P}(V), F(k))$ is a polynomial in k of degree equal to the dimension of support of F.

6. Let $X = \mathbf{P}^n$. Consider the functor $T : \mathcal{M}(\mathcal{O}_X) \to Vect$ given by $T(F) = H^n(X, F)$.

Show that this functor is right exact. Describe a system of objects adapted for this functor and compute its derived functors.

7. Let X be a curve in $\mathbf{P}^2$ defined by a polynomial of degree d.

(i) Suppose X is non-singular. Show how to compute its genus.

(ii) Suppose X is non-singular outside k points and at these points it has simplest nodal singularities.

Compute the arithmetic genus of X. Compute the geometric genus of X, i.e. the genus of its smooth model.

8. Let C be a smooth projective curve. Fix d and consider the variety $S = S^d = C \times C \times ... \times C$ (d times). We have a natural map of sets $p : S \to Div(C)$.

Construct an invertible $\mathcal{O}$-module L on $S \times C$ such that for every $s \in S$ the restriction of L to the fiber $C_s = pr^{-1}(s)$ is canonically isomorphic to $\mathcal{O}(D)$ where $D = p(s)$.

9. Let $\mathcal{A}$ be an abelian category, $C^\cdot, D^\cdot$ two complexes of objects in $\mathcal{A}$.

Define the complex of abelian groups $R^\cdot = Hom^\cdot(C^\cdot, D^\cdot)$ by $R^i =$ morphisms of graded groups $C^\cdot \to D^\cdot$ of degree i.

Show that 0-cycles in complex R are just morphisms of complexes $\nu : C^\cdot \to D^\cdot$. In particular, given any element $h \in R^{-1}$ we get a morphism of complexes $dh : C^\cdot \to D^\cdot$. Such morphisms are called **homotopic to zero** (and element $h \in R^{-1}$ is called a homotopy).

Show that morphisms homotopic to 0 always induce the 0 morphisms on cohomologies. Show that morphisms homotopic to 0 form an ideal in all morphisms of complexes.

10. For any complex $M \in Com(\mathcal{A})$ define its cone $Cone(M) := Cone(Id_M)$. We have a canonical exact sequence of complexes $0 \to M \to Cone(M) \to M[1] \to 0$.

(i) Show that a complex $Cone(M)$ is always acyclic.

(ii) Show that a morphism of complexes $\nu : C \to D$ is homotopic to 0 iff it can be decomposed as $L \to Cone(L) \to M$ and also iff it can be decomposed as $L \to cocone(M) \to M$, where $cocone(M) := Cone(M)[-1]$.