Problem assignment 2.

Representations of reductive p-adic groups.

Joseph Bernstein November 12, 2008.

A remark on problems in different areas. In my assignments I will try to single out problems that are not directly related to representation theory. For example sign (LA) signifies a problem (or a definition) from linear algebra, (Top) from topology and so on.

A remark on different kinds of problems. In all my home assignments I will use the following system.

The problems without marking are just exercises. You have to convince yourself that you can do them but it is not necessary to write them down (if you have difficulties with one of these problems ask me or Dmitry).

The problems marked by [**P**] you should hand in for grading.

The sign $(*)$ marks more difficult problems.

The sign (∇) marks more challenging and more interesting problems which are related to some interesting subjects. They are not always related to the course material, but I definitely advise you to think about these problems.

Definition. **Quasi unital** (for short QU) algebra A is an associative algebra that satisfies

(*) For any finite collection of elements $a_i \in A$ there exists an idempotent $e \in A$ such that $ea_i = a_i e = a_i$ for every i.

If A is QU algebra we call a left A-module M **unital** if $AM = M$. Unless mentioned otherwise all A-modules are assumed to be unital. We denote the category of such modules by $\mathcal{M}(A)$.

1. (i) Let M be a non-zero finitely generated A-module. Show that it has a simple quotient module.

(ii) Show that any non-zero A-module has a simple subquotient.

Notation. We denote by $Irr(A)$ the set of isomorphism classes of simple A-modules. For any A-module M we consider the subset $JH(M) \subset Irr(M)$ of all simple A-modules isomorphic to subquotients of M.

[**P**] **2.** (i) If $L \subset M$ then $JH(L) \bigcup JH(M/L) = JH(M)$

(ii) Suppose M is a sum of submodules M_α. Show that $JH(M) = \bigcup JH(M_\alpha)$.

Definition. Let A be a QU algebra.

Nil radical of A is the maximal two-sided ideal $Nil(A)$ that consists of nilpotent elements (show that such maximal ideal exists).

Jacobson radical $Jac(A)$ is the collection of all elements $a \in A$ that act as 0 in every simple A-module.

3. (i) Show that Jacobson radical always contains the Nil radical.

(ii) Suppose A is an algebra over an algebraically closed field k. Assume that k is uncountable and $\dim_K(A)$ is countable.

Show that in this case Nil radical coincides with the Jacobson radical.

(*) **4.** Let M be an A-module. Show that the following conditions are equivalent

(a) M is isomorphic to a direct sum of simple modules

(b) Every submodule $L \subset M$ has a complementary submodule $N \subset M$ (i.e. $M = L \oplus N$).

Such A-modules M are called **semi-simple** or **completely reducible**.

[P] 5. Consider smooth complex representations of an l-group G.

(i) Show that the Hecke algebra $H(G)$ of locally constant compactly supported distributions on G is a QU algebra and the category $\mathcal{M}(H(G))$ is canonically equivalent to $\mathcal{M}(G)$.

(ii) Show that if G is compact then any representation of G is completely reducible.

(iii) Let $H \subset G$ be an open subgroup of finite index. Show that a representation (π, G, V) is completely reducible iff its restriction to H is completely reducible.

(iv) Show that the Nil radical and the Jacobson radical of the algebra $H(G)$ are trivial.

Definition. Let (π, G, V) be a representation of an l-group, $(\tilde{\pi}, G, \tilde{V})$ its contragradient representation. For every pair of vectors $\xi \in \tilde{V}, v \in V$ we define the matrix coefficient $m_{\xi,v}$ to be a locally constant function on the group G defined by formula $m_{\xi,v}(g) =< \xi, \pi(g)v >$.

[P] 6. Show that the map $(\xi, v) \mapsto m_{\xi,v}$ defines a morphism of representations of the group $G \times G$, namely $m : \tilde{V} \otimes V \to C^{\infty}(G)$.

(*) **7.** Show that a representation π is (quasy) compact iff all its matrix coefficients have compact support.