Problem assignment 3.

Representations of reductive p-adic groups.

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1. Fix a non-trivial character ψ_1 of the group F^+.

Let L be a finite dimensional vector space over F. We consider it as a commutative l-group.

(i) Show how to identify the dual group $\hat{L} = Mor(L, \mathbf{C}^*)$ with the dual space L^* (using the character ψ_1).

(ii) Show that the map $h \mapsto \hat{h}$ defined by $\hat{h}(\psi) =< h, \psi >$ defines an isomorphism of algebras $\mathcal{H}(L) = S(\hat{L})$.

2. (i) Show that for an l-space X the category $Sh(X)$ is canonically equivalent to the category $\mathcal{M}(S(X))$. Describe fibers of the sheaf corresponding to an $S(X)$-module F.

(ii) Give a technical definition of a sheaf F on an l-space X in terms of the collection of the fibers F_x for $x \in X$. In these terms show how to define the pull back functor on sheaves.

3. (i) Describe the set $Irr(F^*)$

Describe the set $Irr(L)$, where L is a finite dimensional F-space.

(ii) Show that the category $\mathcal{M}(L)$ is canonically equivalent to the category of sheaves $Sh(\hat{L})$.

[P] 4. Describe the set $Irr(P_2)$, where $P_2 \subset G_2$ is the mirabolic subgroup.

Hint. Use the extension $1 \to F^+ \to P_2 \to F^* \to 1$.

[P] (*) **5.** Let an l-group G act on an l-space X. Construct a QU algebra $\mathcal{H}(G, X)$ such that the category of equivariant sheaves $Sh_G(X)$ is canonically equivalent to the category $\mathcal{M}(\mathcal{H}(G, X))$.

[P] 6. Let Z be a transitive G-space, $e \in Z$. Consider the stabilizer $H = Stab(e, G)$ of this point. Show that the space Z is canonically isomorphic to G/H.

(i) Show that the category $Sh_G(Z)$ is canonically equivalent to the category $\mathcal{M}(H)$.

(ii) More generally, consider a morphism of G-spaces $p : X \to Z$ and set $X_e = p^{-1}(e)$. Show that the equivariant category $Sh_G(X)$ is canonically equivalent to the category $Sh_H(X_e)$.

7. Let $Z = G/H$ and $e \in Z$ be as in problem 6.

(i) Show that locally constant distributions on Z form an equivariant sheaf. Define a canonical one-dimensional representation $\Delta_{G/H}$ of the group H on the fiber of this sheaf at e.

In particular, for any l-group H we can consider the regular action of the group $G = H \times H$ on the space H. The corresponding representation of H we denote by Δ_H.

[P] (ii) Show that we have a canonical isomorphism $\Delta_{G/H} = \Delta_G \otimes (\Delta_H)^{-1}$.

8. Let G act on an l-space X with finite number of orbits.

(i) Show that all these orbits are locally closed and they form a stratification $\mathcal{S}$ of X.

(ii) Let F be a G-equivariant sheaf on X and $V := S(X, F)$ the space of its sections with compact support.

Show that this is a smooth representation of G.

Show that to every open G-invariant subset $W \subset X$ corresponds a G-invariant subspace V_W.

Show that to every strata S canonically corresponds a subquotient V_S of the representation V.

9. Formulate (and prove) Mackey theorem for l-groups.

10. Let G be an l-group $H \subset G$ a closed subgroup. Consider the induction functor $ind_H^G : \mathcal{M}(H) \to \mathcal{M}(G)$.

Show that this functor is exact.

Show that if the quotient space G/H is compact then the functor $ind_H^G = Ind_H^G$ maps admissible representations into admissible ones.

11. Let ρ be an irreducible admissible representation of the group $G \times H$.

Show that it can be written as a tensor product $\rho = \omega \otimes \sigma$ of admissible irreducible representations of groups G and H.

Show that this decomposition is uniquely defined up to isomorphism. Describe all the possible isomorphisms.