Problem assignment 10.

Algebraic Geometry and Commutative Algebra

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1. Let X be an affine algebraic variety, F a coherent sheaf on X.

(i) Let us assume that support of F has dimension 0.
Show that $\dim \Gamma(X, F) = \sum_{x\in X} \dim F_x$ (sum of dimensions of stalks).

(ii) Suppose X is affine, $\nu : G \to H$ a morphism of coherent sheaves. Suppose that ν is an imbedding and is an isomorphism outside of a subset of dimension 0.
Show that $\dim H(X)/G(X) = \sum_{x\in X} \dim H_x/G_x$

(iii) In particular show that if X is an affine curve and $f \in \mathcal{O}(X)$ a nonzero function, then $\dim \mathcal{O}(X)/f\mathcal{O}(X) = \sum_{x\in X} mult_x(f)$

2. Let C be a smooth curve, F a coherent sheaf on C.
(i) Show that if F does not have torsion then it is locally free.

(ii) Suppose in addition C is affine and $f \in \mathcal{O}(C)$ a nonzero function. Explain how to compute $\dim F(C)/fF(C)$.

3. Let $p : C \to D$ be a dominant morphism of smooth curves. For a given point $d \in D$ set $n(d) := \sum_{c\in p^{-1}(d)} mult_c(p)$.
Show that $n(d)$ does not depend on d. This number n is called the **degree** of morphism p.
Show that degree of p coincides with the degree of the field extension $[k(C) : k(D)]$.

In what follows we fix a smooth projective curve C. We denote by $Div(C)$ the free abelian group generated by points of C. An element $D = \sum_{a\in C} n_a \cdot a$ is called a **divisor** on C. The number $degD = \sum n_a$ is called the **degree** of the divisor D.

Denote by K the field $k(C)$ of rational functions on C. For every function $f \in K^*$ we construct a divisor $div(f) := \sum_{a\in C} Deg_a(f) \cdot a$

4. Check the following facts

(i) The map $deg : Div(C) \to \mathbf{Z}$ is a group homomorphism. It is epimorphism and we denote its kernel by $Div^0(C)$.

(ii) The map $div : K^* \to Div(C)$ is a group homomorphism. Its kernel is the subgroup k^*.
The image of this morphism is called the group of principle divisors (notation $PrinDiv(C)$)

(iii) $deg(div(f)) \equiv 0$. In other words $PrinDiv(C) \subset Div^0(C)$
Important invariants we will study are groups
$Pic(C) := Div(C)/PrinDiv(C)$ and $Pic^0(C) := Div^0(C)/PrinDiv(C)$

Definition. (i) We say that a divisor $D = \sum n_a a$ is effective (or positive) if all coefficients n_a are non-negative. If D, D' are two divisors then the notation $D' \geq D$ means that the divisor $D' - D$ is effective.

(ii) We say that divisors D, D' are equivalent (notation $D' \sim D$) if $D' - D$ is a principle divisor.

Definition. Given a divisor D we denote by $L(D)$ the vector space consisting from functions $f \in K^*$ such that $div(f) + D \geq 0$ and the zero function. We set $l(D) := \dim L(D)$
Show that $L(D)$ is indeed a k-vector subspace in K.

5. Show the following facts
(i) If $D' \sim D$ then $degD' = degD$ and $l(D') = l(D)$
(ii) If $D' \geq D$ then $degD' \geq degD$ and $l(D') \geq l(D)$
(iii) For any point $a \in C$ and any divisor D we have $l(D) \leq l(D + a) \leq l(D) + 1$.
(iv) $l(D) > 0$ iff D is equivalent to an effective divisor.
(v) If $l(D) > 0$ then there exists a point $a \in C$ such that $l(D - a) < l(D)$.

The fundamental problem: given deg(D) find good estimates for the number $l(D)$.

6. Upper bound. Proposition. Let D be a divisor. Show that if $degD < 0$ then $l(D) = 0$. If $degD \geq -1$ then $l(D) \leq degD + 1$

7. Lower bound. Theorem. Set $def(D) = degD + 1 - l(D)$. Show that $def(D)$ is bounded above by some universal constant A that depends only on the curve C. Minimal such constant $g = g(C)$ is called the **genus** of the curve C; it is easy to see that $g(C) \geq 0$.

Hint. (i) Show that the function $def(D)$ depends only on equivalence class of D and is increasing, i.e. if $D' \geq D$ then $def(D') \geq def(D)$.

(ii) Show that there exists a family of divisors $B_n, n \in \mathbf{Z}_+$ such that for every n we have $degB_n \geq n - A_0$ and $def(B_n) \leq A$.

(iii) Given a divisor D show that for large n we have $l(B_N - D) > 0$. From this deduce that $def(D) \leq A$.

8. We will see that an important role plays a function $h(D) := g - def(D) = l(D) + g - 1 - degD$ (in other words $l(D) - h(D) = degD + (1 - g)$).

By definition $h(D) \geq 0$ for all D and there exists a divisor D_{min} such that $h(D_{min}) = 0$.

(i) Show that the function $h(D)$ depends only on equivalence class of D and is decreasing, i.e. $D' \geq D$ implies $h(D') \leq h(D)$.

(ii) Show that there exists a divisor D_0 of degree $g - 1$ such that $h(D_0) = 0$.

(iii) Show that for any divisor D of degree $> 2g - 2$ we have $h(D) = 0$.

Hint. Use the fact that any divisor B of degree $\geq g$ is equivalent to an effective divisor.

9. Let $a \in C$ be an arbitrary point. Consider the following system of divisors $D_k = k \cdot a$, $k \in \mathbf{Z}_+$. We say that the the number k is a **gap** for the point a if $l(D_{k-1}) = l(D_k)$.

(i) Show that there are finite number of gaps for the point a. How many ?

(ii) Show that if we remove from the curve C the point a then the resulting curve C_a is affine.