Problem assignment 14.

Algebraic Geometry and Commutative Algebra

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Definition. (CA) Let A be a commutative algebra (with 1). For any A-module N consider the functor $T_N : \mathcal{M}(A) \to \mathcal{M}(A)$ given by $M \mapsto M \otimes_A N$.

1. (i) Show that the functor $T = T_N$ is right exact.

(ii) Show that functor T has a cohomological derived functor $(L^iT : \mathcal{M}(A) \to \mathcal{M}(A))$ such that $L^iT = 0$ for $i > 0$, $L^0T = T$ and functors L^iT are erasing for $i < 0$.

(iii) Show how to compute the derived functors L^iT using projective (or free) **left** resolutions.

2. Show that there is a natural isomorphism $L^iT_N(M) = L^iT_M(N)$.

Remark. For historical reasons these groups are denoted as $Tor^A_{-i}(M, N)$.

Definition. An A-module N is called **flat** if the functor T_N is exact.

3. (i) Show that free and projective modules are flat. Show that direct sums of flat modules are flat.

(ii) Show that directed direct limits of flat modules are flat. In particular show that localizations of flat modules are flat.

(iii) Show that the tensor product of flat modules is flat.

4. (i) Show that an A-module N is flat iff the functor $L^{-1}T_N$ is 0.

(ii) Suppose A is Noetherian. Show that an A-module N is flat iff the functor T_N is exact on the subcategory of finitely generated A-modules.

Definition. Let $\pi : X \to Y$ be a morphism of affine algebraic varieties. In this case the algebra $B = \mathcal{O}(X)$ is naturally an algebra over $A = \mathcal{O}(Y)$. We say that an $\mathcal{O}_X$-module F is **flat over** Y if the corresponding B-module $F(X)$ is flat over $A = \mathcal{O}(Y)$.

Show that in the situation of this definition $\mathcal{O}(B)$-module F is flat over Y iff for every $a \in X$ the stalk F_a is flat over A (and similarly iff for every $a \in X$ that stalk F_a is flat over the stalk $\mathcal{O}_{Y,b}$ where $b = \pi(a)$).

Definition. Let $\pi : X \to Y$ be a morphism of algebraic varieties, F an $\mathcal{O}_X$-module. We say that F if **flat over** Y if this is true locally.

We say that the morphism π is flat if $\mathcal{O}_X$ is flat over Y.

5. Let $\pi : X \to Y$ be a flat morphism of algebraic varieties. Show that every flat $\mathcal{O}_X$-module F is flat over Y.

We have proven in class that if $\pi : X \to Y$ is a projective morphism of algebraic varieties and F a coherent $\mathcal{O}_X$-module flat over Y then locally on Y there exists a complex $K^{\cdot} = (K^i)$ of free $\mathcal{O}_Y$-modules of finite rank that "computes" the derived functors $R^i\pi_*(F)$ and all of its base changes. This module is defined not uniquely - we ca always replace is by another quasiisomorphic complex of free modules since they will compute the same derived functors. For this reason it is important to analyze such complexes in detail.

6. Let Y be an affine algebraic variety and $K^{\cdot} = (K^i)$ be a complex of free $\mathcal{O}_Y$-modules of finite rank such that $K^i = 0$ for $i < 0$ and for $i > n$. Set $d_i = rkK^i$.

For every point $b \in Y$ consider the fiber complex $K^{\cdot}|_b$ and denote by $a_i = a_i(b)$ sequence of dimensions of cohomologies of this complex.

(i) Show that for every i the function $a_i(b)$ is upper-semicontinuous.

(ii) Suppose that at some point b we have $a_n(b) = 0$. Show that then in some neighborhood of B we can replace the complex $K^{\cdot}$ by quasiisomorphic complex of free modules $L^{\cdot} = (L^i)$ of smaller length (i.e. $L^i = 0$ for $i < 0$ and for $i > n - 1$).

Show that this happens iff the module $H^n(K^{\cdot})$ equals 0 in some neighborhood of the point b.

(iii) More generally, suppose we know that the function a_n is constant near some point b. Show that in some neighborhood of this point we can replace the complex $K^{\cdot}$ by a quasiisomorphic complex $L^{\cdot} \oplus F^{\cdot}$ where $L^{\cdot}$ has smaller length and $F^{\cdot}$ is a complex consisting of one free module F placed in degree n.

Show that this happens iff the module $H^n(K^{\cdot})$ is free in some neighborhood of b.

7. Iterating results of problem 6 analyze what happens in the cases mentioned bellow. Describe how you can replace the complex $K^{\cdot}$ by some more simple quasiisomorphic complex of free modules.

(i) At some point $b \in Y$ we have $a_l(b) = a_{l+1}(b) = ... = a_n(b) = 0$

(ii) At some point $b \in Y$ we know that all the functions $a_l, a_{l+1}, ..., a_n$ have local minima.

8. Let $K^{\cdot}$ be a complex of length N from problem 6.

(i) Show that we always have a base change in the last degree n. This means that for any $\mathcal{O}_Y$-module G we have $H^n(G \otimes K^{\cdot}) = G \otimes H^N(K^{\cdot})$.

(ii) Show that the Euler characteristic $\chi(a) := \sum(-1)^i a_i$ equals to the Euler characteristic $\chi(d)$.

(iii) for every l consider the partial Euler characteristic $\chi_l(a) := \sum_{i \leq l}(-1)^i a_i$. Show that as a function on Y it is upper semi-continues for even l and lower semi-continuous for odd l.

Compare it with the partial Euler characteristic $\chi_l(d)$.

9. Consider the situation described above. We have $\pi : X \to Y$ projective morphism and F coherent $\mathcal{O}_X$-module flat over Y. For every point $b \in Y$ describe the natural morphism $\psi_i(b) : R^i\pi_*(F)|_b \to H^i(X_b, F_b)$.

Let us assume that for some i and some point b this is an epimorphism.

(i) Show that in some neighborhood U of the point b the functor $R^i\pi_*$ commutes with base change.

(ii) Show that the module $R^i\pi_*(F)$ is locally free near the point b iff the morphism $\psi_{i-1}(b)$ is onto.