Problem assignment 15.

Algebraic Geometry and Commutative Algebra

Joseph Bernstein June 11, 2009.

1. Let $i : X \to Y$ be a morphism of separated algebraic varieties. It is called **closed imbedding** if it defines an isomorphism of X with a closed subvariety X_0 of the variety Y.

(i) Show that this is equivalent to the condition that i defines a closed imbedding of topological spaces and the morphism $\mathcal{O}_Y \to i_*(\mathcal{O}_X)$ is epimorphic.

(ii) Let $i : X \to Y$ be a projective morphism. Show that if it has finite fibers than it is a finite morphism.

(iii) Let $i : X \to Y$ be a finite morphism. Fix a point $x \in X$ and set $y = i(x) \in Y$. Let us assume that the fiber $i^{-1}(y)$ consists of one point x.

Show that the morphism i is a closed imbedding of some neighborhood U of the point x into some neighborhood V of the point y iff the differential $Di : T_x(X) \to T_y(Y)$ is a monomorphism.

2. Let X be a projective algebraic variety, $F^{\cdot} = (F^i)$ a finite complex of $\mathcal{O}$-modules on X.

Show that the following conditions are equivalent.

(a) $F^{\cdot}$ is acyclic

(b) For large k the complex of vector spaces $\Gamma(X, F^i(k))$ is acyclic

Definition. Let X be an algebraic variety.

(i) A coherent $\mathcal{O}_X$-module Q is called a **nill-support module** if its support has dimension 0, i.e. it consists of finite number of points. In this case we set $\dim Q := \dim(\Gamma(X, Q))$.

(ii) Let $d \in \mathbf{Z}_+$. A coherent $\mathcal{O}_X$-module F is called d**-separated** if for any quotient nill-support module Q with $\dim Q \leq d + 1$ the natural morphism $\Gamma(X, F) \to \Gamma(X, Q)$ is epimorphic.

Remark. We can also consider slightly more general situation when we are given a coherent $\mathcal{O}_X$-module F together and a finite dimensional vector space V with a morphism $V \to \Gamma(X, F)$. Such pair is called d separated if for any null-support quotient module Q of dimension $\leq d + 1$ the induced morphism $V \to \Gamma(X, Q)$ is epimorphic.

3. (i) Show that F is 0-separated iff it is generated by global sections.

(ii) Let L be an invertible $\mathcal{O}$-module. Show that it is 1 separated iff it is very ample.

(iii) Let N, F be coherent $\mathcal{O}$-modules on X. Suppose N is invertible and is generated by its global sections. Show that if F is d-separated then $N \otimes F$ is also d-separated.

4. Let X be a subvariety of a projective space and F a coherent $\mathcal{O}_X$-module. Show that for any d we can find k such that the twisted module $F(k)$ is d-separated.

Hint. Reduce to the case when X is a projective space and F is a direct sum of $\mathcal{O}$-modules $\mathcal{O}(i)$.

5. Let X be an algebraic variety and N, L invertible $\mathcal{O}$-modules on X.

(i) Suppose we know that L is very ample and that N is generated by global sections. Show that the module $N \otimes L$ is very ample.

(ii) Suppose that L is ample. Show that for large k the $\mathcal{O}$-module $N(k) := N \otimes L^{\otimes k}$ is very ample.