Problem assignment 1.

Representations of p-adic groups - Langlands program.

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A remark on different kinds of problems. In all my home assignments I will use the following system.

The problems without marking are just exercises. You have to convince yourself that you can do them but it is not necessary to write them down (if you have difficulties with one of these problems ask me or Sasha).

The problems marked by [**P**] you should hand in for grading.

The sign $(*)$ marks more difficult problems.

The sign (∇) marks more challenging and more interesting problems which are related to some interesting subjects. They are not always related to the course material, but I definitely advise you to think about these problems.

Definition. Let F be a field. A (multiplicative) **norm** on F is a function $x \in F \mapsto |x| \in \mathbf{R}^+$ that satisfies the following conditions

(0) $|0| = 0, |1| = 1$

(i) $|xy| \equiv |x| \cdot |y|$

(ii) For some constant $C > 0$ we have metric inequality

$|x + y| \leq C(|x| + |y|)$ for all $x, y \in F$.

We call the norm $| \ . \ |$ **non-Archimedean** if it satisfies the ultra-metric inequality $|x + y| \leq \max(|x|, |y|)$.

We say that the norm is trivial if $|x| = 1$ for all $x \in F^*$.

1. Show that the norm is non-Archimedean iff for every natural number n we have $|n \cdot 1| \leq 1$.

Given a norm $| \ . \ |$ on the field F and a real number $R > 0$ we can define a new norm $| \ . \ |'$ by $|x|' = |x|^R$. Such norms we will call **equivalent**.

2. Show that two norms on the field F are equivalent iff they define the same topology on F.

3. Let $(F, | \ . \ |)$ be a normed field. Define the completion $\hat{F}$ of the field F. Show that this is again a normed field and that this field is **complete**.

Explain how to define completion in terms of topology on F.

Definition. Let F be a field. A **place** of F is an equivalence class of non-trivial norms on F (or, equivalently, a non-discrete topology on F that can be defined by a norm).

Given a subfield $k \subset F$ we often will be interested in places of F over k, i.e. places of F that are trivial on k.

4. (i) Let k be an algebraically closed field, $F = k(t)$ the field of rational functions in one variable.

Describe all places of F over k.

(ii) Do the same problem for a field G that is a finite extension of F.

(iii) Do similar problems without assuming that k is algebraically closed. In particular do them for the case of a finite field k.

5. Describe all places of the field $\mathbf{Q}$ (Ostrovsky lemma).

6. Let F be a locally compact field. Using Haar measure define the modulus function $x \mapsto |x|$ on F^* and show that it defines a norm on F.

Show that this norm is trivial iff F is discrete.

7. Let F be a local field, L its finite extension. Show that L is a local field and any norm on F uniquely extends to a norm on L.

8. Classification of local fields.

(i) Show that up to isomorphism there exist just two complete Archimedean fields $\mathbf{R}$ and $\mathbf{C}$.

(ii) Let F be a local non-Archimedean field F.

Show that if $char(F) = 0$ then F a finite extension of one of the fields $\mathbf{Q}_p$.

Show that if $char(F) = p > 0$ then F is a finite extension of the field $\mathbf{F}_p((t))$ (field of Laurent series).

9. Let F be a complete field with a non-Archimedean norm $|\ .\ |$. Define the ring of integers $\mathcal{O}_F$, its maximal ideal $\mathfrak{m} = \mathfrak{m}_F$ and a residue field $k = k_F$. For simplicity we assume that the norm is discrete (i.e. the ideal $\mathfrak{m}$ has s generator $\pi = \pi_F$).

Prove the following statement

Hensel lemma. Let $P \in \mathcal{O}[t]$ be a monic polynomial. $\bar{P} \in k[t]$ its reduction module $\mathfrak{m}$.

Fix a root $y \in k$ of the polynomial $\bar{P}$. Let us assume that it is simple, i.e. the derivative Q of the polynomial $\bar{P}$ does not vanish at y.

Show that there exists unique element $x \in \mathcal{O}$ such $P(x) = 0$ and the reduction $\bar{x} \in k$ equals y.

10. Let F be a local non-Archimedean field. Its residue field k is finite, i.e isomorphic to $\mathbf{F}_q$ for some q.

Consider the polynomial $P = t^q - t \in \mathcal{O}[t]$. Show that the set T of its roots in F lies in $\mathcal{O}$ and that the reduction map $\mathcal{O} \to k$ maps bijectively T onto k. The subset T is called the set of **Teichmuller representatives** of the elements of k.