Problem assignment 1.

Analysis on Manifolds.

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[P] 1. Give a proof to the problem 9 in "Problems in linear algebra".

[P] 2. Sketch the proof of the spectral theorem (problem 10 in "Problems in linear algebra").

[P] 3. Solve problem 8 in "Problems about metric spaces".

[P] 4. Solve problem 9 in "Problems about metric spaces".

Definition. (i) We define a **set with functions** as a pair $(X, S(X))$, where X is a set and $S(X)$ is a subalgebra of the algebra $F(X)$ of all functions on the set X, provided it satisfies the following condition:

(C) The set X is canonically in bijection with the set of all morphisms of $\mathbf{R}$-algebras $Mor_{\mathbf{R}-alg}(C(X), \mathbf{R})$.

(ii) Let $(X, S(X))$ and $(Y, S(Y))$ be two sets with functions. We define a **morphism** $\nu : X \to Y$ to be a map of sets $\nu : X \to Y$ such that $\nu^*(S(Y)) \subset S(X)$.

Equivalent, but more refined, version of this definition is that a morphisms $\nu : X \to Y$ is a pair of a map of sets $\nu : X \to Y$ and a morphism of $\mathbf{R}$-algebras $\nu^* : S(Y) \to S(X)$ that are compatible, i.e. satisfy $f(\nu(x)) \equiv \nu^*(f)(x)$.

[P] 5. Let X be a subset of $\mathbf{R}^n$ with induced metric d. Let $C(X) \subset F(X)$ be the subset of continuous functions (remind that by $F(X)$ we denote the algebra of all real valued functions on X).

(i) Show that $(X, C(X))$ is a set with functions.

(ii) Show that given two sets with functions $(X, C(X))$ and $(Y, C(Y))$ as in (i) the following 3 sets are in natural bijection:

Continuous maps $\nu : X \to Y$

Morphisms of sets with functions $\nu : (X, C(X)) \to (Y, C(Y))$

Morphisms of $\mathbf{R}$-algebras $\nu^* : C(Y) \to C(X)$

Definition. An **abstract domain** of dimension n is a space with functions $(X, S(X))$ that is isomorphic to a set with functions of the form $(D, C^\infty(D))$ for some open subset $D \subset \mathbf{R}^n$.

[P] 6. (i) Show that for any open subset $D \subset \mathbf{R}^n$ the pair $(D, C^\infty(D))$ is a set with functions.

(ii) Let $(X, S(X))$ and $(Y, S(Y))$ be two abstract domains. Show that smooth morphisms $\nu : X \to Y$ are in natural bijection with morphisms of $\mathbf{R}$-algebras $\nu^* : S(Y) \to S(X)$.