Problem assignment 2.

Analysis on Manifolds.

Joseph Bernstein March 5, 2010.

A remark on different kinds of problems. In all my home assignments I will use the following system.

The problems without marking are just exercises. You have to convince yourself that you can do them but it is not necessary to write them down (if you have difficulties with one of these problems ask about it).

The problems marked by [**P**] you should hand in for grading.

The sign $(*)$ marks more difficult problems.

The sign (∇) marks more challenging and more interesting problems which are related to some interesting subjects. They are not always directly related to the course material, but I definitely advise you to think about these problems.

1. Let M be a smooth manifold, $S(M)$ the algebra of smooth functions on M. Every point $a \in M$ defines a morphism of $\mathbf{R}$-algebras $\nu_a : S(M) \to \mathbf{R}$.

We would like to show that under some conditions the map of sets $\nu : M \to Mor_{\mathbf{R}-alg}(S(M), \mathbf{R})$ is a bijection.

[**P**] (i) Assume that M is Hausdorff. Show that then ν is injective.

[**P**] (ii) Show that if M is compact then ν is a bijection.

(*) (iii) Assume that M is Hausdorff and second-countable (i.e. it has a countable base of open subsets).

Show that in this case there exists an exhaustive function $f \in S(M)$, i.e. f satisfies the following property:

(*) For every $T \in \mathbf{R}$ the set $M_T = \{x \in M | \ f(x) \leq T\}$ is compact.

Using the function f show that the map ν is surjective.

We will assume that all manifold are Hausdorff and second-countable.

[**P**] **2.** (i) Show that for any two manifolds M, N we have $Mor(M, N) = Mor_{\mathbf{R}-alg}(S(N), S(M))$.

(ii) Show that for any manifold M we have natural isomorphism $Vect(M) \simeq Der_{\mathbf{R}}(S(M))$.

(iii) Show that for any manifold M we have natural isomorphism $\Omega^1(M) \simeq Mor_{S(M)}(Vect(M), S(M))$.

iv) Show that there exists unique linear operator $d : \Omega(M) \to \Omega(M)$ of degree 1 that satisfies the Leibnitz rule, equation $d^2 = 0$ and coincides with the standard DeRham operator on $\Omega^0(M) = S(M)$.

3. Let (x^i) be a coordinate system on a domain D. Show how to write a covector field α in this coordinate system.

Let (y^j) be another coordinate system on D. Explain how to write coefficients of α in coordinates y if we know them in coordinates x.

Do the same for vector fields, for differential forms.

4. For any vector field ξ define the inner multiplication $i_\xi : \Omega(M) \to \Omega(M)$. Define the operator $L_\xi : \Omega(M) \to \Omega(M)$ by $L_\xi = [d, i_\xi]$ (pay attention to signs).

Show Weyl formulas for these operators:

(i) d, i_ξ, L_ξ are derivations of the algebra $\Omega(M)$.
(ii) $d^2 = 0$; $i_\xi^2 = 0$ and hence $[i_\xi, i_\eta] = 0$.
(iii) $[d, L_\xi] = 0$ and on $\Omega^0(M)$ operator L_ξ coincides with ξ.
(iv) $[L_\xi, i_\eta] = i_{[\xi,\eta]}$, $[L_\xi, L_\eta] = L_{[\xi,\eta]}$

5. Let $\nu : M \to N$ be a morphisms of smooth manifolds, $a \in M$ and $b = \nu(a) \in N$. Set $m = \dim M, n = \dim N$.

[P] (i) Suppose ν is submersion at a, i.e. the differential $D\nu_a$ is onto. Show that for any local coordinate system y on N we can find a local coordinate system (x) on M such that $\nu(x_1, ..., x_m) = (x_1, ..., x_n)$ (in this case $m \geq n$).

[P] (ii) Suppose ν is immersion at a, i.e. the differential $D\nu_a$ is injective. Show that for any local coordinate system x on M we can find a local coordinate system (y) on N such that $\nu(x_1, ..., x_m) = (x_1, ..., x_m, 0, ..., 0)$ (in this case $m \leq n$).

(*) (iii)(**Constant rank Theorem**). Suppose that the rank of the operator $D\nu_x$ equals k for all points x close to a. Show that we can find local coordinate systems (x) on M and (y) on N such that $\nu(x_1, ..., x_k, x_{k+1}, ..., x_m) = (x_1, ..., x_k, 0, ..., 0)$ (in this case $k \leq m$ and $k \leq n$).

[P] 6. Let M be a smooth manifold and $f_1, ..., f_k \in S(M)$ and $Z \subset M$ is the set of common zeroes of functions f_i. We say that functions f_i are independent at a point $a \in Z$ if their differentials are linearly independent at this point.

Show that in this case Z is a submanifold of M near the point a. Describe its tangent space at a.