**Problem assignment 3.**

Analysis on Manifolds.

Joseph Bernstein March 14, 2010.

**1.** Let $M$ be a manifold and $\omega \in \Omega(M)$. We would like to express the DeRham differential $d\omega$ in terms independent of coordinate systems.

(i) Show that for 1-form $\alpha$ we have $d\alpha(\xi,\eta) = \xi(\alpha(\eta)) - \eta(\alpha(\xi)) + \alpha([\xi,\eta])$

Show that for a $k$-form $\omega$ we have

$d\omega(\xi_1,...,\xi_{k+1}) = \sum_{1\leq i\leq k+1}(-1)^{i+1}\xi_i(\omega(\xi_1,...,\hat{\xi_i},...\xi_{k+1}))+$
$+\sum_{1\leq i<j\leq k+1}(-1)^{i+j+1}\omega([\xi_i,\xi_j],\xi_1,...,\hat{\xi_i},...,\hat{\xi_j},...,\xi_{k+1})$

**Hint.** Use Weyl formulas.

Let $\nu : N \to S$ be a morphism of manifolds, $Z \subset S$ be a submanifold. We would like to study the subset $W = \nu^{-1}(Z) \subset N$ near some point $a \in W$.

**2.** (i) Suppose $\nu$ is a submersion at the point $a$. Show that then $W$ is a submanifold near $a$ and compute its tangent space.

(ii) Let us make a weaker assumption that at the point $b = \nu(a)$ the sum of subspaces $T_bZ$ and $D\nu(T_aN)$ equals to $T_bS$ (in this case we say that $\nu$ is **transversal** to the submanifold $Z$ at the point a). Solve the same problem as in (i) for the transversal case.

**3.** More generally, consider two morphisms of manifolds $\mu : M \to S$ and $\nu : N \to S$. We define the **fiber product** $W \subset M \times N$ to be the set of points $(m,n)$ such that $\mu(m) = \nu(n)$.

We say that the morphisms $\mu, \nu$ are **transversal** at the point $(m,n) \in W$ if the sum of tangent spaces $D\mu(T_mM)$ and $D\nu(T_nN)$ equals to the tangent space $T_cS$ at the point $c = \mu(m) = \nu(n)$.

Show that if $(m,n) \in W$ is a point of transversality, then the subset $W$ is a submanifold near this point. Compute its tangent space.

**4.** Let $M$ be a manifold, $V$ a vector space.

Define the space $\Omega(M,V)$ of differential forms on $M$ with values in $V$. Define the following structures on this space:

(i) Structure of left $\Omega(M)$-module.

(ii) For any vector field $\xi$ on $M$ define the inner product $i_\xi : \Omega(M,V) \to \Omega(M,V)$.

(iii) Define the DeRham differential $d : \Omega(M,V) \to \Omega(M,V)$.

(iv) Define Lie derivatives $L_\xi$.

(v) Show that all the Weyl's formulas work in this case.

**5.** Choose a moving frame $e_1,...,e_n$ in $V$. Then we can write $\Omega(M,V) = \oplus_i \Omega(M)e_i$.

(i) Explain how all operations are written in this frame. Show that all these operations are described by a matrix $\omega_{ij}$ of 1-forms. Write consistency relation for these forms.

(ii) Suppose additionally we are given a morphism $\nu : M \to V$. Write $D\nu = \sum \omega_i e_i$. Compute formulas for $d\omega_i$.

**6.** Let us assume that the space $V$ in problem 4 is Euclidean, i.e. it is equipped with a scalar product $< , >$.

Define the scalar product $< , >: \Omega(M, V) \times \Omega(M, V) \to \Omega(M)$.

(i) Show that $L_\xi(<\omega, \rho>) = <L_\xi(\omega), \rho> + <\omega, L_\xi(\rho)>$

(ii) Write all relations between forms $\omega_{ij}$ and $\omega_i$ in this case.

**7.** Let $M$ be a surface $\nu : M \to V = \mathbf{R}^3$ an imbedding. Choose a Darboux frame $(e_1, e_2, e_3)$. This means that we have chosen orthonormal vector fields $\xi_1, \xi_2$ on $M$, consider the corresponding functions $e_i = D\nu(\xi_i)$ from $M$ to $V$ and extend them to an orthonormal frame by adding the orthogonal function $e_3$.

Consider the $2 \times 2$ matrix $h_{ij} = \omega_{i3}(e_j)$.

(i) Show that it equals $<de_3(\xi_i), e_j> = <\xi_i(e_3), e_j>$

(ii) Show that this is equal to $-\xi_i(\xi_j(f(x, a)))|_{x=a}$, where $f(x, a) = <\nu(x), e_3(a)>$. In particular show that this is twice the matrix of the second fundamental form $Q_{II}$ in the basis $(\xi_i)$.

**8.** Write a formula for the Gauss curvature $G$ in terms of the form $\omega_{12}$. Using this write the formula for the Gauss curvature in terms of the basis of 1-forms $\omega_1, \omega_2$ dual to the basis of vector fields $\xi_1, \xi_2$.