Problem assignment 6.

Analysis on Manifolds.

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Definition. Fix manifolds M, N; we assume M to be compact. Let $\mathcal{P}$ be some property of morphisms of manifolds $\nu : M \to N$. We say that the property $\mathcal{P}$ is **stable** if for any smooth family of morphisms $\nu : S \times M \to N$ the set $S_{\mathcal{P}}$ of all points $s \in S$ such that the morphism ν_s has property $\mathcal{P}$ is open in S.

1. (i) Show that the following properties are not stable: (a) topological imbedding; (b) homeomorphism (c) epimorphism

(ii) Show that immersion and submersion are stable properties. Show that manifold imbedding is a stable property. In particular show that diffeomorphism is a stable property.

2. Let $\nu : M \to N$ be a morphism of manifolds. Show that we can model it on the canonical projection $pr : \mathbf{R}^p \to \mathbf{R}^{p+q}$ for some p, q, i.e. we can realize M and N as closed submanifolds in these Euclidean spaces such that ν is the restriction of the projection.

Give some reasonable upper bound for p and q in terms of dimensions of M and N.

3. Show that any manifold N of dimension k can be realized as a closed submanifold of the space $\mathbf{R}^n$ with $n = 4k + 5$.

Show that in fact one can choose realization with $n = 2k + 1$.

4. (i) Let U be an abstract domain and $(x), (y)$ two coordinate systems on U. Show that a subset $A \subset U$. has measure 0 with respect to the system (x) iff it has measure 0 Suppose with respect to (y).

(ii) Let A be a subset of a manifold M. Show that it has measure 0 iff it locally has measure 0.

5. Let $\nu : R \to N$ be a morphism of manifolds transversal to a submanifold $Z \subset N$. We denote by $W = W(\nu, Z)$ the submanifold $\nu^{-1}(Z) \subset R$.

Similarly consider another morphism $\nu' : R \to N'$ and define the submanifold $W' \subset R$. Show that the restriction of the morphism ν to W' is transversal to Z iff the restriction of ν' to W is transversal to Z'.

6. Let $N \subset \mathbf{R}^l$ be a closed submanifold. We say that a point $a \in \mathbf{R}^l$ has a good projection to N if the function $Q_a(x) = ||x - a||^2$ has unique minimum on N at some point $x(a)$.

Show that any point $b \in N$ has a neighborhood U in $\mathbf{R}^l$ such that all points $a \in U$ have good projection on N and the map $p : U \to N$ defined by $p : a \mapsto x(a)$ is a smooth submersive morphism.

Deduce that there exist a neighborhood V of N in $\mathbf{R}^l$ such that the projection $p : V \to N$ is well defined, smooth and submersive.