Problem assignment 7.

Analysis on Manifolds.

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1. Give the definition of the **transversality** of two morphisms $\nu : M \to N$ and $\nu' : M' \to N$.

Show that for transversal morphisms the fibered product $W = M \times_N M'$ is a manifold of expected dimension. Explain how to compute tangent spaces of W.

2. Let $p : R \to N$ be a morphism transversal to a submanifold $Z \subset N$; consider the submanifold $W = \nu^{-1}(Z)$. Similarly consider a morphism $p' : R \to N'$ and submanifold $W' = \nu'^{-1}(Z')$.

Show that the restriction of p to W' is transversal to Z iff the restriction of p' to W is transversal to Z'.

3. Let $M \subset N$ be a compact submanifold. Denote by R the normal vector bundle to M in N.

Show that there exists a neighborhood U of M in N that is isomorphic to a neighborhood of zero section in the total space $Tot(R)$ of this vector bundle. In particular show the existence of the projection $U \to M$.

4. Show that the relation of homotopy of morphisms is an equivalence relation.

Hint. Show that any homotopy can be transferred into a homotopy constant near the ends.

5. Let $\nu : M \to N$ be a morphism of manifolds with M compact. Show that any morphism ν' close to ν is homotopic to ν.

6. Given a morphism $\nu_0 : M \to N$ consider two smooth families of its deformations ν_s, $\nu_{s'}$ parameterized by manifolds S, S'.

Show that there exists a larger family of deformations λ parameterized by some manifold T such that families ν and ν' are obtained from it by restricting to submanifolds $S, S' \subset T$.

Hint. Prove this first for $N = \mathbf{R}$.

7. Consider a morphism $\nu_0 : M \to N$ and a submanifold $Z \subset N$; we assume M to be compact. Suppose we found some closed subset $C \subset M$ such that ν_0 is transversal to Z in some neighborhood of C.

Show that there exists a deformation of ν_0 (i.e. a family of morphisms $\nu_s : M \to N$) that is generically transversal to Z and is constant (i.e coincides with ν_0) near the subset C.

8. Consider the half-space $H = H^n$ and decompose it into interior part H_0 and boundary ∂H.

Show that neighborhoods of points in H_0 and in ∂H are not diffeomorphic.

9. Let M be a manifold with boundary. Show that there exists a smooth non-negative function f on M such that 0 is its regular value and $f^{-1}(0) = \partial M$.

10. Let M be a manifold of dimension k. Show that it can be realized as a closed submanifold of an Euclidean space $\mathbf{R}^n$ for some n. In fact show that one can take $n = 2k + 1$.

Hint. Follow the following steps:

(i) Construct exhaustive function p on M (i.e. a proper morphism $M \to \mathbf{R}$).

(ii) Using (i) show that it is enough to construct a collection of functions $(f_i |\ i = 1, ..., n)$ that satisfy the following condition

(*) Functions f_i separate points and their differentials separate all tangent vectors.

(iii) Show that for any compact subset $C \subset M$ one can find functions $(f_i |\ i = 1, ..., 2k + 1)$ that satisfy condition (*) on C.

(iv) For every $k \in \mathbf{Z}$ consider a compact subset $C_k = p^{-1}([k, k+1]) \subset M$. Set $R = \bigcup_{k-even} C_k$. Construct a family of functions $(f_i |\ i = 1, ..., 2k + 2)$ that satisfy the condition (*) on R.

Similarly construct the family of functions for the subset $R' = \bigcup_{k-odd} C_k$.

(v) Show that one can realize M as a closed submanifold in $\mathbf{R}^{2k+1}$.