Analysis on Manifolds

Syllabus.

Preliminary material.

Reminder of basic notions of linear algebra over $\mathbf{R}$ and $\mathbf{C}$.

Linear space, dual space, endomorphism space, group of automorphisms.

Basis, dimension, coordinates.

Quadratic forms. Euclidean spaces.

Metric and topology on the Euclidean space $\mathbf{R}^n$.

Basic notions of metric spaces and topological spaces. Local and global properties.

Compact spaces and their properties.

Differentiable functions on a domain $D \subset \mathbf{R}^n$. Smooth functions on subsets of $\mathbf{R}^n$. Differentiable morphisms.

Basic notions of manifold theory.

Notion of a manifold. Smooth submanifolds of $\mathbf{R}^n$ – naive manifolds.

Abstract domains. Spaces with functions. Definition of abstract domains and abstract manifolds. Morphisms of manifolds. Local coordinate systems.

Inverse and implicit function theorems.

Local to global correspondence.

Cut-off functions and partitions of unity.

Notions of tangent and cotangent spaces.

Differentials of morphisms. Differentials of functions.

Vector fields. Commutator of vector fields.

Exterior differential calculus.

Differential forms. De Rham differential. Functoriality. Lie derivative. Weyl formulas.

DeRham complex. General remarks on homologies and cohomologies. DeRham cohomology. Poincare lemma and the DeRham theorem.

Classical theory of surfaces in Euclidean 3-space.

First and second fundamental forms. Principle curvatures. Gauss map and Gauss curvature.

Proof of Gauss theorem on intrinsic character of Gauss curvature using the method of moving frames.

Theory of vector bundles

Vector bundles (real and complex). Examples. Morphisms of vector bundles. Subbundles, quotient bundles. Metrics on bundles.

Inverse image of vector bundles.

Connections on a vector bundle E. Curvature of a connection. Structure of the space of connections.

Invariants of vector bundles: Chern classes, Euler characteristic.

Vector bundles and principle bundles. Geometric description of connections.

Lie groups and torsers.

Notion of a Lie group. Examples. G-torsers (principle bundles with a group G). Reduction of the structure group. Extension of torsers.

Torsers and vector bundles.

Connections on torsers. Algebraic description of connections. Geometric description of connections. Curvature of the connection.

Invariants of torsers.

Transversality theory.

Notion of transversality. Basic properties.

Sard's Lemma.

Applications of Sard lemma.

Deformation theorem and moving lemma. Morse functions. Embedding of manifolds into Euclidean space. Index of a morphism.

Basic notions of Riemannian geometry.

Notion of a Riemannian manifold. Riemannian metric. Geodesics.

22. Affine connections. Torsion and curvature. Geodesics of an affine connection. Exponential morphism.

Levi-Civita connection. Riemannian geodesics as extremal curves. Existence of geodesics.

Possible additional topics.

Morse theory.

Symplectic geometry.

Complex geometry.