Problem assignment 2.

Super Geometry and applications in Physics.

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Construction and description of super manifolds.

We will discuss a general method of constructing and describing super manifolds that is essentially due to Grothendieck.

Definition. A super manifold (or s-manifold for short) is a pair $\mathcal{M} = (M, \mathcal{O})$, where M is topological space and $\mathcal{O}$ a sheaf of superalgebras on M, that satisfies the axioms bellow.

Sections of the sheaf $\mathcal{O}$ on an open subset $U \subset M$ we call **smooth functions on** U . The pair $\mathcal{U} = (U, \mathcal{O})$ we call an open submanifold of $\mathcal{M}$. We denote the super algebra of smooth sections on U by $S(\mathcal{U})$.

Here are the axioms of s-manifold:

(1) The space M is Hausdorff with countable base of open subsets.

(2) Locally $\mathcal{M}$ is isomorphic to a super domain $\mathcal{D}$.

We will use the following notations

Man - category of manifolds, *sMan*-category of s-manifolds, *Dom* - category of domains, *sDom* - category of s-domains.

Our main tool for describing s-manifolds are contravariant functors $F : sMan^0 \to Sets$. Geometrically one should think about a set $F(B)$ as a set of B-points of some s-manifold $\mathcal{M}$ that we are trying to describe (i.e. the set morphisms of s-manifolds $\nu : B \to \mathcal{M}$); equivalently this is the set of families of points of $\mathcal{M}$ parameterized by a base B.

We denote by *sFunct* the category of such functors. The basic result that we use is the following general categorical result

Yoneda lemma. Consider the natural covariant functor $i : sMan \to sFunct$ given by $\mathcal{M} \mapsto F_{\mathcal{M}}$, where $F_{\mathcal{M}}(B) := Mor(B, \mathcal{M})$. This functor i is fully faithful.

So, following the idea of Grothendieck, we will employ the following strategy for constructing (and describing) s-manifolds.

In order to construct an s-manifold we construct a functor $F \in sFunct$. We say that the functor F is **representable** if there exists an s-manifold $\mathcal{M}$ and an isomorphism of functors $\kappa : F \to F_{\mathcal{M}}$.

Note that if such pair exists then the s-manifold $\mathcal{M}$ is defined up to **unique** isomorphism, i.e. it is well defined by this construction.

Usually most of technical work in this approach is going to a **proof** that a given functor F that we constructed is representable. This usually is much easier to do than to make the same proofs inside constructions, which is done in usual approaches to construction of interesting objects.

1. Let V be a linear s-space of dimension (m, n). Consider the functor F defined by $F(B) := (S(B) \otimes V)_{\bar{0}}$.

Show that this functor is representable. We will denote the corresponding s-manifold by $\mathbf{V}$. It is a super space of dimension (m, n) (it is isomorphic to $\mathcal{R}^{(m,n)}$.

Consider an arbitrary functor $F \in sFunct$. We would like to describe constructions related to F that in the case of a representable functor $F_{\mathcal{M}}$ would give some geometric objects associated to $\mathcal{M}$.

2. (i) Describe a set $\bar{F}$ that corresponds to the set of points.

(ii) Describe a super algebra $S(F)$ that corresponds to the algebra of functions. For every point $a \in \bar{F}$ describe the evaluation morphism $\nu_a : S(F) \to \mathbf{R}$.

(iii) Describe a topology on the set $\bar{F}$.

(iv) To every open subset $U \subset \bar{F}$ assign a subfunctor $F^U \subset F$.

3. Prove the following general result

Theorem. Let $F \in sFunct$ be a functor with the following properties:

(i) The topological space $\bar{F}$ is Hausdorff with countable base of open subsets
(ii) Locally on $\bar{F}$ the functor F is representable.

Show that in this case the functor F is representable.

4. Corollary. Suppose we found an s-manifold $\mathcal{M}$ and a morphism of functors $p : F \to F_{\mathcal{M}}$ that satisfy the following conditions.

(*) The s-manifold $\mathcal{M}$ has a covering by open subsets U such that the functors $p^{-1}(U) \subset F$ are representable (here you should describe for every open subset $U \subset \mathcal{M}$ a subfunctor $p^{-1}(U) \subset F$ – its preimage in F).

Show that then the functor F is representable.

5. Let $\mathcal{M}$ be an s-manifold and E a vector bundle on $\mathcal{M}$ (i.e. E is a locally free sheaf of $\mathcal{O}$-modules on M of finite rank (p, q)).

Define the s-manifold $Tot(E)$ that is "the total space of E".

Hint. First define it as a functor and then show that this functor is representable.

6. Let $\mathcal{G}$ be an s-manifold. Show that to define a structure of a Lie s-group on $\mathcal{G}$ is the same as to define compatible structures of groups on sets $F(B)$, where $F = F_{\mathcal{G}}$.

Similarly for an action of a Lie s-group on an s-manifold.

7. Let V be an s-space of dimension (m, n). Describe the Lie s-group $\mathcal{G} = GL(V)$. (Write it in coordinates, describe the algebra $S(\mathcal{G})$).

Definition. Let γ be an action of a Lie s-group $\mathcal{G}$ on an s-manifold $\mathcal{M}$. Important role is played by the morphism $R_\gamma : \mathcal{G} \times \mathcal{M} \to \mathcal{M} \times \mathcal{M}$.

(i) The action γ is called **transitive** if the morphism R_γ is a surjective submersion.

(ii) The action γ is called **free** if the morphism R_γ is an immersion with closed image.

8. Proposition. Suppose γ is a free action of a Lie s-group $\mathcal{G}$ on an s-manifold $\mathcal{M}$. Then there exists a quotient s-manifold $\mathcal{G}\backslash\mathcal{M}$.

9. Proposition. Let γ be a transitive action of a Lie s-group $\mathcal{G}$ on an s-manifold $\mathcal{M}$. Fix a point $a \in \mathcal{M}$.

Define the stabilizer $\mathcal{H} \subset \mathcal{G}$ of the point a. Show that this is a subgroup and that $\mathcal{M}$ is isomorphic to the quotient s-manifold $Q = \mathcal{G}/\mathcal{H}$.

10. Let V be a super space of dimension (m, n). Define the s-manifold $Gr_{(k,l)}(V)$ of subspaces of V of dimension (k, l). (First define it as a functor and then show that this functor is representable).