**Problem assignment 2.**

Algebraic Geometry and Commutative Algebra

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**A remark on problems in different areas.** In my assignments I will try to single out problems that are not directly related to algebraic geometry. For example sign (CA) signifies a problem (or a definition) from commutative algebra, (LA) stands for linear algebra, (Top) for topology.

**A remark on different kinds of problems.** In all my home assignments I will use the following system.

The problems without marking are just exercises. You have to convince yourself that you can do them but it is not necessary to write them down (if you have difficulties with one of these problems ask me or Sasha).

The problems marked by [**P**] you should hand in for grading.

The sign $(*)$ marks more difficult problems.

The sign $(\nabla)$ marks more challenging and more interesting problems which are related to some interesting subjects. They are not always directly needed in the course, but I definitely advise you to think about these problems.

**Remark.** In this assignment you can freely use the Nullstellensatz and Serre's lemma but you should specify when you use them. For an algebraic variety $X$ we denote by $\mathcal{O}(X)$ the algebra of global regular functions on $X$.

**Definition**. An **algebraic variety** is a space with a sheaf of functions $(X, \mathcal{T}(X), \mathcal{O}(X))$ that satisfies the following condition

(*) There exists a finite covering of $X$ by open subsets $U_i$ such that for every $i$ the space with a sheaf of functions $\mathbf{U}_i$ obtained by restriction of structures to set $U_i$ is isomorphic to some affine algebraic variety.

Intuitively this means that $X$ is glued from affine algebraic varieties $U_i$.

**[P] 1.** Let $X$ be an algebraic variety. Fix a finite open covering of $X$ consisting of affine algebraic varieties $B_i$.

Show that a function $\phi$ on $X$ is regular iff its restriction to every subset $B_i$ is polynomial.

Show how using the covering $(B_i)$ one can give an explicit description of the space of regular functions $\mathcal{O}(X)$ as the kernel of a morphism $\nu : \oplus_i \ \mathcal{P}(B_i) \to \oplus_{i,j} \ \mathcal{P}(B_i \bigcap B_j)$ (here we assume that all the intersections $B_i \bigcap B_j$ are affine - this will be true for most of interesting cases).

**[P] 2.** Fix a finite dimensional vector space $V$ over $k$.

(i) Describe the corresponding algebraic variety (affine space) $\mathbf{V}$ whose set of points is $V$. Describe explicitly the algebra $\mathcal{O}(\mathbf{V})$.

(ii) Let $f$ be a coordinate function on $V$ (i.e. a non-zero linear function). Consider basic open subvariety $\mathbf{V}_f \subset \mathbf{V}$. Show that $V_f$ is an affine algebraic variety and that it is isomorphic to $H \times k^*$, where $H$ is the hyperplane given by equation $f(x) = 1$. Describe explicitly $\mathcal{O}(\mathbf{V}_f)$.

(iii) Denote by $\mathbf{V}^*$ an open algebraic subvariety $\mathbf{V}^* = \mathbf{V} \setminus 0 \subset \mathbf{V}$. Describe the algebra $\mathcal{O}(\mathbf{V}^*)$.

**Hint.** Do this first for the case $n = 1$, then $n = 2$, then the general case.

**The quotient construction.** Let $(X, \mathcal{T}_X, \mathcal{O}_X)$ be a space with a sheaf of functions. Let $p : X \to Y$ be an epimorphic map of sets. Show how in this case one can canonically define on $Y$ the structure of a space with sheaf of functions.

**3.** In problem 2 let $X = P(V)$ be the set of one-dimensional subspaces of $V$ (projective space). Consider the natural projection of sets $p : \mathbf{V}^* \to X$.

(i) Using the quotient construction define topology $\mathcal{T}(X)$ and a sheaf of functions $\mathcal{O}_X$ on the set $X$.

Show that this is an algebraic variety. More precisely show that for any coordinate function $f$ the image $p(\mathbf{V}_f)$ is an open subset of $X$ isomorphic to an affine space (as a space with a sheaf of functions).

We will denote this variety by $\mathbf{P}(V)$. In case when $V$ is the standard coordinate space with coordinates $(t_o, ..., t_n)$ this variety is usually denoted by $\mathbf{P}^n$.

**[P] 4.** (i) Describe the algebra $\mathcal{O}(\mathbf{P}^n)$ of global regular functions on $\mathbf{P}^n$.

(ii) Let $X$ be an algebraic variety obtained from $\mathbf{P}^2$ by removing one point. Describe the algebra $\mathcal{O}(X)$ of global regular functions on $X$.

**Definition**. (Top) Let $X$ be a topological space. A subset $Y \subset X$ is called **locally closed** if it satisfies the following equivalent conditions

(i) $Y$ is an intersection of an open and a closed subsets of $X$.

(ii) $Y$ is locally closed, i.e. any point $x \in Y$ has an open neighborhood $U$ such that $U \cap Y$ is closed in $U$ .

(iii) $Y$ is open in $cl(Y)$ (here $cl(A)$ is the closure of $A$).

Check that these conditions are equivalent.

**[P] 5.** Let $X$ be an algebraic variety and $Z \subset X$ a locally closed subset. Show that $Z$ has a canonical structure of an algebraic variety.

Here you should use the following finiteness lemma that we will prove soon

**Lemma.** Let $X$ be an affine algebraic variety and let $U$ be any open subset of $X$. Then $U$ can be covered by a finite number of basic open subsets $X_f$.

**[P] 6.** Let $V$ be a vector space and $X = \mathbf{P}(V)$ the corresponding projective space (see the problem 2). We would like to describe the algebraic structure of this space in more detail. Later we will use this description many times.

Denote by $A$ the algebra of polynomial functions on $V$. This is a graded algebra $A = \oplus A^k$.

Given a homogeneous polynomial $f \in A^k$ with $k > 0$ we consider the corresponding basic open subset $V_f \subset \mathbf{V}$ and denote by $X_f$ its image in the projective space $X$.

(i) Show that the sets $X_f$ form a basis of the Zariski topology on $X$.

(ii) Show that every subset $X_f$ is an affine algebraic variety and the algebra of regular functions $\mathcal{O}(X_f)$ is isomorphic to the subalgebra $A^0_f$ of functions of degree 0 in the graded algebra $A_f$.

In order to do this you will need a result from linear algebra described in the next problem.

**Definition**. (LA) Let $H$ be a group. By definition a **character** of $H$ is a homomorphism $\chi : H \to k^*$.

Suppose we fixed an action of the group $H$ on a $k$-vector space $L$. For any character $\chi$ of $H$ we consider eigen subspace

$L^\chi := \{v \in L |\ hv = \chi(h)v \text{ for all } h \in H\}$

**[P] 7.** (LA). Let a group $H$ act on a space $L$. Fix a collection of characters $\chi_1, ..., \chi_m$ of $H$ pairwise distinct.

Suppose we have an equality in $L$ of the form $v = \sum_i v_i$, where $v_i \in L^{\chi_i}$. Show that every vector $v_i$ can be written as a linear combination of vectors $hv$ for some elements $h \in H$.

In particular show that if $v = 0$ then all vectors $v_i$ are 0.

(*) **8.** Let $V$ be an $n$-dimensional $k$-vector space. Fix a number $l \leq n$ and denote by $Gr_l(V)$ the set of all subspaces $L \subset V$ of dimension $l$.

Describe the natural structure of an algebraic variety on this set. This variety is called the **Grassmann variety** or **Grassmannian**.

**Hint.** First understand that in the space of matrices $Mat(l, n)$ the subset of matrices of any given rank $r$ is an algebraic subvariety.

$\nabla$ **9.** Let $A$ be a finitely generated $k$-algebra (commutative, with 1). Consider the set $M(A) := \mathrm{Mor}_{k-alg}(A, k)$.

(i) Describe Zariski topology on the set $M(A)$. Show that the set $M(A)$ has a natural structure of an affine algebraic variety. In particular describe the algebra of regular functions $\mathcal{O}(M(A))$.

(ii) Given a morphism of $k$-algebras $\mu : B \to A$ show that the corresponding map $\nu : M(A) \to M(B)$ is a morphism of affine algebraic varieties.

Give an example of different morphisms of algebras that give the same morphism of varieties.