Algebraic Geometry and Commutative Algebra

Problem assignment 5.

Joseph Bernstein April 4, 2011.

[P] 1. Let $\nu : X \to Y$ be a morphism of algebraic varieties.

(i) Show that if ν is **dominant**, i.e. its image is dense in Y, then $\dim X \geq \dim Y$.

(ii) Show that if ν has finite fibers then $\dim X \leq \dim Y$.

(iii) Show that if ν has finite fibers then their cardinalities are bounded by some constant C_ν.

∇ **2.** Let X be an affine algebraic variety. Fix a finite dimensional subspace $V \subset \mathcal{P}(X)$ and consider the vector space $D = V \oplus V \oplus ... \oplus V$ (sum of d copies). Every point $d - (v_1, ..., v_d) \in D$ defines a morphism ν_d of the algebra $B = k[y_1, ..., y_d]$ into $\mathcal{P}(X)$ by $\nu_d(y_i = v_i$.

(i) Consider the subset $U \subset D$ of points $d \in D$ such that the morphism ν_d is finite. Show that this is an open subset. In particular if it is not empty then it is dense.

(ii) Let us define the dimension $\dim_F(X)$ of an affine algebraic variety X to be the minimal number d for which there exists a finite morphism $\nu : B = k[y_1, ..., y_d] \to \mathcal{P}(X)$.

Using (i) show that this function satisfies the basic properties of dimension function:

(a) For open covering $X = \bigcup U_i$ we have $\dim X = \max \dim U_i$

(b) For finite epimorphism $\nu : X \to Y$ we have $\dim X = \dim Y$.

(c) $\dim \mathbf{A}^d = d$

[P] 3. (i) Let us consider subset $M_r(m, n)$ of the space of $m \times n$ matrices consisting of all matrices of rank r. Show that this is an algebraic variety and compute its dimension.

(ii) Compute dimension of the Grassmannian manifold $Gr_k(V)$ -variety of k-dimensional subspaces of V, where $\dim V = n$.

(iii) Compute dimension of the set of all quadratic hypersurfaces in $\mathbf{P}^6$.

[P] 4. (i) Let $X \subset \mathbf{P}^6$ be a closed surface (i.e. 2-dimensional subvariety). Let us denote by L_X the set of all lines $\mathbf{P}^1 \subset \mathbf{P}^6$ intersecting X.

Show that L_X is a closed algebraic subvariety of the space $L(\mathbf{P}^6)$ of all lines in $\mathbf{P}^6$ and compute its dimension.

(ii) Let X, Y, Z be three closed surfaces in $\mathbf{P}^6$. We are looking for lines $l \subset \mathbf{P}^6$ that intersect all three of them. Show that if there exists such a line then there are infinitely many such lines.

Definition. A function u on a topological space X is called **constructible** if it takes finite number of values and every level set of it is constructible.

The general ideology of algebraic geometry is that if u is a function on an algebraic variety X with integer values which is "algebraically defined", then it is always constructible.

Let X be a variety over a base S, i.e. it is given together with a morphism $p_X : X \to S$ of algebraic varieties. We interpret S as a base and consider the family of varieties $X_s := (p_X)^{-1}(s)$ for $s \in S$ as an "algebraic family of varieties" parameterized by points of S.

Similarly, given two varieties X, Y_1 over S and a morphism $\nu : X \to Y$ over S (formulate precisely what it means) we get an "algebraic family of morphisms $\nu_s : X_s \to Y_s$ parameterized by points of the base S.

We would like to consider natural functions and properties depending on points of $s \in S$ which describe some algebraic properties of varieties X_s and morphisms ν_s.

[P] 5. (i) Consider the function $u(s) := \dim X_s$. Show that u is a constructible function on S.

(*) (iii) Show that if ν has finite fibers then the function $u(s) = \sharp(X_s)$ is constructible.

[P] (*) **6.** Consider the situation in problem 5 and assume for simplicity that S, X, Y are affine. Given some property $\mathbf{P}$ of morphisms of algebraic varieties let us consider the subset $S_{\mathbf{P}} \subset S$ consisting of points s such that the morphism ν_s satisfies $\mathbf{P}$.

Show that the subsets $S_{\mathbf{P}}$ are constructible for the following properties:

(i) imbedding (ii) closed imbedding
(iii) epimorphism (iv) dominant morphism
(v) finite morphism (vi) morphism with finite fibers

∇ (*) (*) **7.** In problem 5 consider the function c on the base S defined as follows $c(s) :=$ number of irredicible components of X_s.

Show that this function is constructible (I do not know how to prove this though this is definitely correct).

[P] ∇ **8.** (CA) Let A be a ring (commutative with 1). Denote by $Spec(A)$ the set of its prime ideals.

(i) Introduce the Zariski topology on $Spec(A)$.

(ii) Show that the intersection of all prime ideals equals to the Nil radical of A.

(iii) Suppose we know that the ring A is Noetherian.

Show that A has a finite number of minimal prime ideals and that every prime ideal of A contains some minimal prime ideal. In particular, show that the intersection of the minimal prime ideals of A equals to the Nil radical of A.

Hint. Show that $Spec(A)$ is a Noetherian topological space and study its irreducible components.