Problem assignment 7.

Algebraic Geometry and Commutative Algebra

Joseph Bernstein June 2, 2011.

[P] 1. Compute $\mathbf{Z}/m\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}/n\mathbf{Z}$.

[P] 2. Let A be a commutative ring. Consider an A-linear functor $T : \mathcal{M}(A) \to \mathcal{M}(A)$. Let us assume that this functor is strongly right exact. Show that then T is isomorphic to the functor T_M for some A-module M (here $T_M(N) = M \otimes_A N$).

3. Let A be a commutative ring A and S a subset of A.

(i) Show that the localization functor $M \mapsto M_S$ is strictly exact.

(ii) Show that $(M \otimes_A N)_S = M_S \otimes_A N = M \otimes_A N_S = M_S \otimes_{A_S} N_S$ (here = everywhere means canonical isomorphism).

In other words, the tensor product commutes with localization.

For two A-modules M, N consider a new A-module $Hom_A(M, N)$. This is a functor contravariant in M and covariant in N.

Let us fix M and study the functor $H_M : \mathcal{M}(A) \to \mathcal{M}(A)$ given by $H_M(N) := Hom_A(M, N)$

4. (i) Show that the functor H_M is left exact

(ii) Show that if M is finitely generated then the functor H_M commutes with localization and arbitrary direct sums.

(iii) Let X be an algebraic variety and let F, G be two $\mathcal{O}$-modules on X. Show that if F is coherent then we can define a new $\mathcal{O}$-module $\mathcal{H}om(F, G)$ (it is called inner hom).

Show that the space of global sections $\Gamma(X, \mathcal{H}om(F, G))$ is naturally isomorphic to the space $Hom(F, G)$ of global morphisms between F and G.

5. Let B be an A-algebra, where A and B are commutative algebras with 1. Show that the restriction functor $Res : \mathcal{M}(B) \to \mathcal{M}(A)$ has left adjoin functor T. Describe this functor.

6. Let $F : A \to B$ and $G : B \to A$ be additive functors between abelian categories. Suppose that F is left adjoint to G.

(i) Show that F is right exact and commutes with infinite direct sums (and more generally with arbitrary direct limits).

(ii) Show that G is left exact and commutes with infinite direct products.

Remark. Usually when you have a right exact functor F which commutes with direct sums you can expect that it admits a right adjoint functor.

7. Let X be a topological space. Show that the natural inclusion functor $i : Sh(X) \to Presh(X)$ has left adjoint functor.

Describe this functor explicitly.

Definition. Let A be a commutative algebra. An A-module M is called **flat** if the functor $T_M : N \mapsto M \otimes_A N$ is exact.

[P] 8. Let M be a flat A-module and S a subset of A. Show that the localized module M_S is flat.

Definition. An A-module P is called **projective** if the functor $H_P : N \mapsto Hom_A(P, N)$ is exact.

[P] 9. (i) Show that an A-module P is projective iff it is isomorphic to a direct summand of a free module.

(ii) Show that P is projective and finitely generated iff it is isomorphic to a direct summand of a free finitely generated module.

(iii) Show that any projective A-module P is flat.

[P] 10. Let X be an affine algebraic variety and $A = \mathcal{O}(X)$.

Let P be a finitely generated A-module. Show that the following three conditions are equivalent:

a) P is projective

b) P is locally free

c) P is flat