Problem assignment 1.

Functions of Complex variables, II

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A remark on problems in different areas. In my assignments I will try to single out problems that are not directly related to complex analysis. For example sign (LA) stands for problems or notions of linear algebra, (Top) for topology.

A remark on different kinds of problems. In all my home assignments I will use the following system.

The problems without marking are just exercises. You have to convince yourself that you can do them but it is not necessary to write them down (if you have difficulties with one of these problems ask me or Jiuzu).

The problems marked by [**P**] you should hand in for grading.

The sign $(*)$ marks more difficult problems.

The sign (∇) marks more challenging and more interesting problems which are related to some interesting subjects. They are not always directly needed in the course, but I definitely advise you to think about these problems.

1. Consider properties of a power series $S = \sum_{n=0}^{\infty} a_n z^n$, $a_n \in \mathbf{C}$.

(i) Show that there exists unique real number $R \in [0, \infty]$ such that the series $S(z)$ converges for $|z| < R$ and diverges for $|z| > R$. This number is called **the radius of convergence** of series S (R might be 0, any positive number or ∞).

Show that $1/R = \limsup |a_n|^{1/n}$

(ii) Show that for $|z| > R$ the sequence of terms of the series S is unbounded.

Let C be a compact subset of the open disc D_R of radius R. Show that the series $S(z)$ converges absolutely and uniformly for all $z \in C$. In particular it converges to a continuous function f on the disc D_R.

(iii) Show that the series $T = \sum a_n z^{n+1}$ has the same radius of convergence R and converges on D_R to the function zf.

Show that the series $S' = \sum n a_n z^{n-1}$ has the same radius of convergence R and converges on D_R to the function $\frac{df}{dz}$.

2. Let f be a continuous function on some convex domain Ω. Show that the following conditions are equivalent:

(i) $\int_{\partial T} f(z)dz = 0$ for any triangle $T \subset \Omega$.

(ii) There exists a differentiable function F on D such that $f = \frac{dF}{dz}$.

(iii) $\int_{\gamma} f(z)dz = 0$ for any nice closed curve $\gamma \subset \Omega$.

(iii) For any two nice pathes γ, δ connecting points a, b inside the domain Ω we have $\int_{\gamma} f(z)dz = \int_{\delta} f(z)dz$.

3. Suppose that function f in problem 2 is differentiable. Show that then it satisfies the equivalent properties of problem 2.

Sketch. Proof by contradiction. Suppose there exists a triangle T such that $\int_{\partial T} f(z)dz = I$, where $I \neq 0$. Let us denote by l its diameter $diam(T)$.

Construct by induction a sequence of imbedded triangles $T = T_0 \supset T_1 \supset ...$ such that $diam(T_k) \leq \frac{l}{2^k}$ and $\int_{\partial T_k} f(z)dz = I_k$ where $|I_k| \geq |I|/4^k$.

Let a be a common point of all this triangles. Using differentiability of the function f at point a show that for small triangles R around a the value J of the integral $\int_{\partial R} f(z)dz$ is $o((diam R)^2)$. This leads to a contradiction.

4. Convince yourselves that you can deduce the following corollaries

(i) **Cauchy formula.** Let Ω be a domain such that $\bar{\Omega}$ is compact and has nice boundary curve $\partial\Omega$. Let us orient $\partial\Omega$ in such a way that Ω is always on the left. Let f be a continuous function f on $\bar{\Omega}$ that is differentiable on Ω. Then $\int_{\partial\Omega} f(z)dz = 0$.

(ii) **Residue formula.** Suppose conditions in (i) are satisfied except for finite number of points $x_1, ...x_n \in \Omega$. Then $\int_{\partial\Omega} f(z)dz = 2\pi i \cdot \sum Res_{x_i} f$.

(iii) **Cauchy integral representation.** Assume conditions in (i). Then for any point $a \in \Omega$ we have $f(a) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{z-a} dz$

5. Show that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

6. Let $D = D_L(0)$ denote the open disc of radius L around 0. Suppose we are given a continuous function f on D that is differentiable.

(i) Show that there exists a power series $S = \sum_{n=0}^{\infty} a_n z^n$ that converges to f at all points of D.

(iii) Show that power series with this property is uniquely defined. Give formulas for coefficients of this series. Show that radius of convergence of this series is $\geq L$.

7. Consider a continuous function f on a disc $D = D_R(0)$. Use the results above to show that the following conditions are equivalent:

(i) f is holomorphic.

(ii) $\int_{\partial T} f(z)dz = 0$ for any triangle $T \subset D$.

(iii) f is a sum of some power series S that converges at all points of D.

8. Let us study **Laurent series** $S(z) = \sum_{n=-\infty}^{\infty} a_n z^n$.

Show that there exist numbers $0 \leq r \leq R \leq \infty$ such that the series $S(z)$ converges for $r < |z| < R$ and diverges if $|z| < r$ or $|z| > R$.

Show that this series converges to a holomorphic (i.e. differentiable) function f on the annulus $A = A(r, R) = \{z | \ r < |z| < R\}$.

Conversely for any annulus $A = A(r, R)$, where $0 \leq r < R \leq \infty$, and any holomorphic function f on A show that there exists a **unique** Laurent series that converges to f on A. Write down explicitly formulas for coefficients of this series.

9. Fix a domain Ω. Let $f_i(z)$ be a sequence of holomorphic functions onΩ. Suppose that $\sum_i f_i(z)$ is converges locally uniformly on Ω to some function $F(z)$. Show that this function is holomorphic.

In fact we will freely use the following more general statement

Principle. Let C be a compact and μ some measure on C that is convergent. Fix a domain $\Omega \subset \mathbf{C}$ and consider a function $f(z, x)$ on $\Omega \times C$. Suppose we know that f is continuous and for every $x \in C$ it is holomorphic in z. Then the function $F(z) := \int_C f(z, x) d\mu$ is holomorphic.

Prove this using criterion in problem 7 in cases where you understand how to define the integral (e.g assume that C is a cube in $\mathbf{R}^n$ and μ is given by integration with a function on C that has absolutely convergent integral).