Problem assignment 2.

Functions of Complex variables, II

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A remark on problems in different areas. In my assignments I will try to single out problems that are not directly related to complex analysis. For example sign (LA) stands for problems or notions of linear algebra, (Top) for topology.

A remark on different kinds of problems. In all my home assignments I will use the following system.

The problems without marking are just exercises. You have to convince yourself that you can do them but it is not necessary to write them down (if you have difficulties with one of these problems ask me or Jiuzu).

The problems marked by [**P**] you should hand in for grading.

The sign $(*)$ marks more difficult problems.

The sign (∇) marks more challenging and more interesting problems which are related to some interesting subjects. They are not always directly needed in the course, but I definitely advise you to think about these problems.

[**P**] **1.** (i) Let ϕ be a continuous function on the line $\mathbf{R}$ that is rapidly decreasing at plus infinity and is smooth in some neighborhood of 0. Using meromorphic continuation define a function $I_\phi(z) := \int_{x>0} \phi(x)x^z dx$. Show that it is meromorphic function and describe its possible poles.

(ii) Consider Euler Gamma function Γ defined by $\Gamma(z) = \int_0^\infty x^z e^{-x} \frac{dx}{x}$.

Show that it is a meromorphic function of z and prove the recurrence functional equation $\Gamma(z+1) = z\Gamma(z)$. Describe all the poles of Gamma function and compute values and poles of this function at all integral points.

∇ **2.** Define the integral $\delta(z) = \int_0^\infty x^z e^{ix} \frac{dx}{x}$. Prove that this is a meromorphic function in z and write a functional equation for this function. Express this function in terms of the Gamma function.

[**P**] **3.** (Top). Let X be a topological space.

(i) Show that if X is path connected then it is connected.

(ii) Show that if X is connected and is locally path connected then it is path connected.

∇ (iii) Construct an example of a space that is path connected but not locally path connected.

4. Let X be a path connected topological space. It is called **simply connected** if any loop in X can be homotopically contracted to a point.

(i) Show that this is equivalent to the following condition

(*) For every two points $a, b \in X$ every two pathes γ, δ from a to b are homotopic.

∇ (ii) Give an example of a simply connected space that is not locally simply connected.

[**P**] **5.** Let Ω be a domain in $\mathbf{C}$ and f a holomorphic function on Ω.

(i) Consider two nice pathes γ, δ from a to b. Show that if γ and δ are homotopic then $\int_\gamma f(z)dz = \int_\delta f(z)dz$.

(ii) Show that if the domain Ω is simply connected then the function f has anti-derivative on Ω, i.e. a function F such that $\frac{dF}{dz} \equiv f$ (in Lang it is called primitive of f).

∇ (iii) Show that one can uniquely define the integral $\int_\gamma f(z)dz$ for all continuous pathes from a to b in such a way that it depends only on the homotopy type if the path is nice then it is given by actual integration.

6. Show that $\mathbf{C} \setminus \mathbf{R}_{\geq 0}$ is simply connected and $\mathbf{C} \setminus 0$ is not simply connected.

[P] 7. Consider a power series $S(z) = \sum_{n\geq 0} a_n z^n$. Let us assume that its radius of convergence R is ≥ 1.

Let us denote by f the holomorphic function in the unit disc D given by this series. We are interested in possible behavior of this function near the boundary circle $S = \partial D$.

(i) Construct examples when f can not be extended to a continuous function on the closed disc $\bar{D}$. If f can be extended we denote the corresponding function on S by u_f (it is called the boundary value of f).

Write condition for a continuous function u on S to be a boundary value of some function f as above.

(ii) Let us assume that radius of convergence is equal 1. Construct example of boundary value function u that is continuous. Construct example when u is five times differentiable. Construct example when u is smooth (i.e. C^∞).

(iii) Show that if the boundary value function is everywhere analytic on S then convergence radius R is strictly larger than 1.

[P] 8. Let f be an entire function.

(i) Suppose we know that $|f(z)| \leq C(1+|z|)^N$ for some constants C, N. Show that f is a polynomial.

(ii) Suppose that the function f is algebraic, i.e. it satisfies an equation $\sum_{i=0}^N P_i(z)f(z))^i$ for some collection of polynomials P_i, not all of them zero. Show that then f is a polynomial function.

(*) **9.** Let f be an entire function. Suppose we know that f is periodic with period 1 and $|f(z)| \leq Cexp(|z|)$. Show that f is constant.

What conclusion about f you can make if it is periodic with period 1 and is bounded by $C\exp(10|z|)$.

[P] 10. Lindelöf theorem. Consider a strip $S = \{(x+yi)|x \geq 0, |y| \leq 1\}$. Let f be a continuous function on S that is holomorphic in the interior of S. Suppose we know that on the boundary ∂S the function $|f(z)|$ is bounded by 1. We would like to show that $|f(z)| \leq 1$ for all $z \in S$.

(i) Show that this is not correct without additional assumptions of f.

(ii) Suppose we found a holomorphic function H on S continuous to the boundary such that its real part $R(z) = Re(H(z))$ satisfies the following conditions

(α) R is positive

(β) $\limsup \frac{\ln(|f(z)|)}{R(Z)} \leq 0$ when $z \to \infty$

Show that under this conditions $|f(z)|$ is bounded by 1 on S. Check that in condition α it is enough to assume that R is positive outside of some compact subset.

Work out example of the function $H(z) = \exp(Cz)$. where C is a constant between 0 and $\pi/4$. Write down explicitly what kind o f bounds it gives on growth of f that ensure that $|f|$ is bounded by 1.