Problem assignment 4.

Functions of Complex variables, II

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1.. Let f be a continuous function on the closed disc $\bar{D}$ of radius R holomorphic on D. We assume that f is not zero at 0 and at all points of the boundary circle $S = \partial D$. Let $I_1(f) := Av_S \log(|f(z)|)$ and $I_0(f) := \log(|f(0)|)$.

Prove **Jensen formula.** $I_1(f) - I_0(f) = \sum_a \log(\frac{R}{a})$,

where the sum is taken over all the zeroes of f inside D (with multiplicities).

Hint. Can assume $R = 1$. Show that functionals I_1, I_0 and sum over zeroes are multiplicative (i.e. $I(fh) \equiv I(f) + I(h)$). Show that the formula holds for the case when there are no zeroes, since in this case the function $\log(|f(z)|)$ is harmonic. Show that if $a \in D$ then the formula holds for the function f_a defined by $f_a(z) = \frac{z-a}{1-\bar{a}z} = z\ \frac{z-a}{z^{-1}-\bar{a}}$ since $|f_a(z)| \equiv 1$ on the unit circle. Show that any f can be written as a product of functions f_a and a function without zeroes.

Let $\mathcal{R}$ be a sequence of non-negative numbers $r_1 \leq r_2 \leq ...$ going to ∞. We define a counting function $N = N_{\mathcal{R}}$ by $N(x) =$ number of terms r_i that are $\leq x$.

Given a number $\rho \geq 0$ we say that the sequence (r_i) **has growth** $\leq \rho$ if one of two equivalent conditions hold

(i) For any $\lambda > \rho$ we have $N(x) \leq C(\lambda)\ x^{\lambda}$ for large x.

(ii) For any $\lambda > \rho$ the sum $\sum r_i^{-\lambda}$ is convergent (here we sum only terms with $r_i \neq 0$).

If Z is a discrete subset of $\mathbf{C}$ (with finite multiplicities) we order elements of Z so that $r_i = |z_i|$ increase and define the **growth** of Z to be infimum of all numbers ρ such that the sequence r_i has growth $\leq \rho$.

[P] 2. Check that conditions (i), (ii) above are equivalent.

Definition. Fix a number $\rho \geq 0$. Let f be a non-zero entire function. We say that f is of order $\leq \rho$ if for any $\lambda > \rho$ we have an estimate $\log(|f(z)|) \leq C|z|^{\lambda}$ for large $|z|$.

[P] 3. Let f be a non-zero entire function of order $\leq \rho$. Using Jensen formula how that the set Z of its zeroes has growth $\leq \rho$.

Fix a number $\rho \geq 0$. We fix natural number k so that $k - 1 \leq \rho < k$. We will also usually fix a number λ such that $k - 1 \leq \rho < \lambda < k$.

We would like to give estimates for Weierstrass products.

Consider holomorphic function

$E_k(w) = (1 - w)\exp(w + w^2/2 + ... + w^{k-1}/(k-1))$

and denote by $g_k(w)$ the real valued function $g_k(w) := \log|E_k(w)|$.

4. Prove the following bounds

(i) Estimate near 0

(C) $|E_k(w) - 1| \leq C|w|^{\lambda}$ if $|w| < 1/2$

(ii) Global upper bound.

(Up) $g_k(w) \leq C_k\ |w|^{\lambda}$

(iii) (Absolute value bounds)

Let $t < 1/2$ be a positive number. Consider discs $D_{1/2} \supset D_t$ centered at 1 of radiuses 1/2 and t.

Then we have the following bounds

(A1) $|g_k(w)| \leq C|w|^{\lambda}$ if w is outside $D_{1/2}$.

(A2) $|g_k(w)| \leq C + |\log(t)|$ for w inside $D_{1/2}$ but outside D_t.

Let $Z = (z_i)$ be a discrete subset of $\mathbf{C}$ of growth $\leq \rho$. For simplicity we assume that Z does not contain 0. Consider the Weierstrass product

$f(z) = \prod E_k(z/z_i)$ and set $g(z) = \log|f(z)| = \sum g_k(z/z_i)$.

[P] 5. Prove the following upper bound estimates

(i) The product absolutely converges everywhere

(ii) $g(z) \leq C|z|^\lambda$

[P] 6. Set $t_i = r_i^{-\lambda-1}$. Consider the domain U equal to the union of discs $D_i = zi \cdot D_{t_i} = \{z|\ z/z_i \in D_{t_i}\}$.

Show that outside U for large $|z|$ we have an estimate $|g(z)| \leq C|z|^{\lambda+\varepsilon}$

Hint. Summing absolute value bounds we get that for every z with $|z| = R$ the value $g(z)$ can be written as a sum of two terms g' and g'', where g' is sum of terms $g_k(z/z_i)$ when the point $w_i = z/z_i$ does not lie in disc $D_{1/2}$ around point 1 and g'' is the sum of terms when w_i lies in disc $D_{1/2}$.

Note that in first case (A1) gives us a bound $|g_k(w_i)| \leq C|w_i|^\lambda$ so we have a bound $|g'| \leq C|z|^\lambda \sum |z_i|^{-\lambda}$.

In the second sum we have only finite number of summands and the corresponding points z_i satisfy $|z_i| \leq 2R$. This implies that there are $\leq CR^\lambda$ terms in this sum; according to estimate(A2) every term is bounded by $C \log R < CR^\varepsilon$.

7. Show that if we consider the projection $p : \mathbf{C} \to \mathbf{R}$ given by $z \mapsto |z|$ then the image of the domain U has measure $\leq C$. In particular for every radius R we can find a radius R' not far from R such that the circle $S_{R'}$ of radius R' around 0 does not intersect U.

[P] 8. Let ϕ be an entire function of order $\leq \rho$. We assume that $\phi(0) \neq 0$. Let $Z = (z_i)$ denote its set of zeroes $Z = Z(\phi)$.

(i) Show that the growth of the set Z is bounded by ρ.

(ii) Using the set Z and k as above consider Weierstrass product $f(z) = \prod E_k(z/z_i)$.

Show that $\phi(z) = f(z)\exp(P(z))$, where P is a polynomial function of degree $\leq \rho$.

(iii) Consider an "exceptional" set $U = \bigcup D_i$, where D_i is a disc centered at z_i of radius $|z_i|^{-\lambda}$.

Show that outside of the set U we have a bound $|\phi(z)| \geq \exp(-C|z|^{\lambda+\varepsilon})$.

(iv) Show that the exceptional set U is small, namely that the projection of the set U to $\mathbf{R}$ has finite measure.

Remark. This holds for any $\lambda > \rho$ and $\varepsilon > 0$.

[P] 9. Fix a number $\rho \geq 0$. Let $E = E_\rho$ denote the space of all entire functions f of order $\leq \rho$. Show that this is an algebra without zero divisors.

We denote by $K = K_\rho$ the field of meromorphic functions generated by this algebra. If a meromorphic function f lies in K_ρ we say that f has order $\leq \rho$.

(i) Describe explicitly the multiplicative group E^* of invertible elements in the algebra E.

(ii) Describe explicitly the quotient group K^*/E^*.

(iii) Let f be a non-zero meromorphic function. We would like to know whether it is of order $\leq \rho$. Denote by Z and P sets of zeroes and poles of the function f.

Show that f lies in K_ρ iff it satisfies the following conditions

(*) Set P has growth $\leq \rho$.

(**) For any $\mu > \lambda > \rho$ consider exceptional set U equal to the union of discs D_q centered at points $q \in P$, where radius of D_q equals $\min(1, |q|^{-\lambda})$. Then outside of U we have a bound $|f(z)| \leq \exp(C\ (1 + |z|)^\mu)$.