Problem assignment 10.

Algebraic Geometry and Commutative Algebra

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[P] 1. Let X be an irreducible separated algebraic variety, $K = k(X)$ its field of rational functions.

For every point $x \in X$ consider the local algebra $\mathcal{O}_{x,X}$ as a subalgebra of K. Show that for different points $x, y \in X$ these subalgebras are different.

2. (i) Show that any curve C is a quasi-projective variety, i.e. it can be realized as a subvariety of a projective space.

Hint. Use Chow's lemma.

(ii) Show that a smooth curve C can be realized as subvariety of $\mathbf{P}^3$.

(iii) Construct a curve C that can not be realized as a subvariety of $\mathbf{P}^{1000}$

3. Let C be a smooth curve, F a coherent sheaf on C.

(i) Show that if F does not have torsion then it is locally free.

(ii) Suppose in addition C is affine and $f \in \mathcal{O}(C)$ a nonzero function. Explain how to compute $\dim F(C)/fF(C)$.

4. Let $p : C \to D$ be a dominant morphism of smooth projective curves. For a given point $d \in D$ set $n(d) := \sum_{c \in p^{-1}(d)} mult_c(p)$.

Show that $n(d)$ does not depend on d. This number n is called the **degree** of morphism p.

Show that degree of p coincides with the degree of the field extension $[k(C) : k(D)]$.

In what follows we fix a smooth projective curve C. We denote by $Div(C)$ the free abelian group generated by points of C. An element $D = \sum_{a \in C} n_a \cdot a$ is called a **divisor** on C. The number $degD = \sum n_a$ is called the **degree** of the divisor D.

Denote by K the field $k(C)$ of rational functions on C. For every function $f \in K^*$ we construct a divisor $div(f) := \sum_{a \in C} deg_a(f) \cdot a$

5. Check the following facts

(i) The map $deg : Div(C) \to \mathbf{Z}$ is a group homomorphism. It is epimorphism and we denote its kernel by $Div^0(C)$.

(ii) The map $div : K^* \to Div(C)$ is a group homomorphism. Its kernel is the subgroup k^*.

The image of this morphism is called the group of principle divisors (notation $PrinDiv(C)$)

(iii) $deg(div(f)) \equiv 0$. In other words $PrinDiv(C) \subset Div^0(C)$

Important invariant that we are going to study is the **Picard group** $Pic(C)$ defined by $Pic(C) := Div(C)/PrinDiv(C)$.

We also consider its subgroup $Pic^0(C) := Div^0(C)/PrinDiv(C)$.

Definition. (i) We say that a divisor $D = \sum n_a a$ is effective (or positive) if all coefficients n_a are non-negative. If D, D' are two divisors then the notation $D' \geq D$ means that the divisor $D' - D$ is effective.

(ii) We say that divisors D, D' are equivalent (notation $D' \sim D$) if $D' - D$ is a principle divisor.

Definition. Given a divisor D we denote by $L(D)$ the vector space consisting from functions $f \in K^*$ such that $div(f) + D \geq 0$ and the zero function. We set $l(D) := \dim L(D)$

Show that $L(D)$ is indeed a k-vector subspace in K.

[P] 6. Show the following facts
(i) If $D' \sim D$ then $degD' = degD$ and $l(D') = l(D)$
(ii) $l(D) > 0$ iff D is equivalent to an effective divisor.
(iii) For any point $a \in C$ and any divisor D we have $l(D) \leq l(D+a) \leq l(D) + 1$.
(iv) If $l(D) > 0$ then for almost every point $a \in C$ we have $l(D-a) = l(D) - 1$.

The fundamental problem: given a divisor D find good estimates for the number $l(D)$.

7. Upper bound. Proposition. Let D be a divisor. Show that if $degD < 0$ then $l(D) = 0$. If $degD \geq -1$ then $l(D) \leq degD + 1$

Definition. For any divisor D we set $\mathrm{def}(D) = degD + 1 - l(D)$ (we call this **defect** of D).

8. Lower bound. Theorem. Show that $\mathrm{def}(D)$ is bounded above by some universal constant A that depends only on the curve C. Minimal such constant $g = g(C)$ is called the **genus** of the curve C; show that $g(C) \geq 0$.

Hint. (i) Show that the function $\mathrm{def}(D)$ depends only on equivalence class of D and is increasing, i.e. if $D' \geq D$ then $\mathrm{def}(D') \geq \mathrm{def}(D)$.

(ii) Show that there exists a family of divisors $D_k, k \in \mathbf{Z}_+$, such that degrees of D_k tend to ∞ and defects of D_k are bounded by some constant A.

(iii) Given a divisor D show that for large k we have $l(D_k - D) > 0$. From this deduce that $\mathrm{def}(D) \leq A$.

Definition. Important role in what follows plays a function
$h(D) := g - \mathrm{def}(D) = l(D) + g - 1 - degD$ (equivalently $l(D) - h(D) = degD + 1 - g$).
By definition $h(D) \geq 0$ for all D and there exists a divisor D_{min} such that $h(D_{min}) = 0$.

[P] 9. (i) Show that the function $h(D)$ depends only on equivalence class of D and is decreasing. More precisely, for any point $a \in C$ we have $h(D) \geq h(D+a) \geq h(D) - 1$.

(ii) Show that there exists a divisor D_0 of degree $g-1$ such that $h(D_0) = 0$.

(iii) Let D be a divisor of degree $> 2g-2$. Show that $h(D) = 0$. Compute $l(D)$.

Hint. Use the fact that any divisor B of degree $\geq g$ is equivalent to an effective divisor.

[P] 10. Let $a \in C$ be an arbitrary point. Consider the following system of divisors $D_k = k \cdot a$, $k \in \mathbf{Z}_+$. We say that the the number k is a **gap** for the point a if $l(D_{k-1}) = l(D_k)$.

(i) Show that there exists a finite number of gaps for the point a. How many ?

(ii) Show that if we remove from the curve C the point a then the resulting curve C_a is affine.