Problem assignment 15.

Algebraic Geometry and Commutative Algebra 2.

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1. Let $i : X \to Y$ be a morphism of separated algebraic varieties. We say that i is a **closed imbedding of algebraic varieties** if it defines an isomorphism of X with a closed subvariety X_0 of the variety Y.

(i) Show that this is equivalent to the condition that i defines a closed imbedding of topological spaces and the morphism $\mathcal{O}_Y \to i_*(\mathcal{O}_X)$ is epimorphic.

(ii) Let $i : X \to Y$ be a projective morphism. Show that if it has finite fibers then it is a finite morphism.

(iii) Let $i : X \to Y$ be a finite morphism. Fix a point $x \in X$ and set $y = i(x) \in Y$. Let us assume that the fiber $i^{-1}(y)$ consists of one point x.

Show that the morphism i is closed imbedding of algebraic varieties in some neighborhood U of the point x into some neighborhood V of the point y iff its differential $D(i) : T_x(X) \to T_y(Y)$ is a monomorphism.

2. Let $F^{\cdot} = (F^i)$ a finite complex of coherent $\mathcal{O}$-modules on a projective space X.

Show that the following conditions are equivalent.

(a) $F^{\cdot}$ is acyclic.

(b) For large k the complex of vector spaces $\Gamma(X, F^i(k))$ is acyclic.

Definition. Let X be an algebraic variety.

(i) A coherent $\mathcal{O}_X$-module Q is called a **nill-support module** if its support has dimension 0, i.e. it consists of finite number of points. In this case we set $\dim Q := \dim(\Gamma(X, Q))$.

(ii) Let $d \in \mathbf{Z}_+$. A coherent $\mathcal{O}_X$-module F is called d**-separated** if for any quotient nill-support module Q with $\dim Q \leq d + 1$ the natural morphism $\Gamma(X, F) \to \Gamma(X, Q)$ is epimorphic.

Remark. We can also consider slightly more general situation when we are given a coherent $\mathcal{O}_X$-module F together and a finite dimensional vector space V with a morphism $V \to \Gamma(X, F)$. Such pair is called d separated if for any null-support quotient module Q of dimension $\leq d + 1$ the induced morphism $V \to \Gamma(X, Q)$ is epimorphic.

3. (i) Show that F is 0-separated iff it is generated by global sections.

(ii) Let X be a projective variety and L be an invertible $\mathcal{O}_X$-module. Show that it is 1 separated iff it is very ample.

(Remind that L is called very ample if its sections define a closed imbedding of the algebraic variety X into a projective space).

(iii) Let N, F be coherent $\mathcal{O}$-modules on X. Suppose N is invertible and is generated by its global sections. Show that if F is d-separated then $N \otimes F$ is also d-separated.

4. Let X be a subvariety of a projective space and F a coherent $\mathcal{O}_X$-module. Show that for any d we can find k such that the twisted module $F(k)$ is d-separated.

Hint. Reduce to the case when F is a direct sum of $\mathcal{O}$-modules $\mathcal{O}(i)$.

5. Let X be a projective algebraic variety and N, L invertible $\mathcal{O}$-modules on X.

(i) Suppose we know that L is very ample and that N is generated by global sections. Show that the module $N \otimes L$ is very ample.

(ii) Suppose that L is ample (that means that some power of L is very ample). Show that for large k the $\mathcal{O}$-module $N(k) := N \otimes L^{\otimes k}$ is very ample.

(iii) Show that for a projective variety X the group $Pic(X)$ is generated by very ample invertible $\mathcal{O}$-modules.

6. Consider the natural projection $p : X = \mathbf{P}^n \times Y \to Y$. We can think about it as a constant family of projective spaces X_y parameterized by points $y \in Y$.

(i) Fix a coherent $\mathcal{O}$-module F on X and consider the corresponding family of coherent sheaves F_y on $X_y = \mathbf{P}^n$.

Show that there exists a number k_0 such that for every $k > k_0$ and every $y \in Y$ the twisted sheaf $F_y(k)$ is acyclic.

(ii) More generally, show that one can choose k_0 so that for any $\mathcal{O}$-module G on Y and any $k > k_0$ the sheaf $F(k) \otimes p^*(G)$ is p_*-acyclic.

7. Suppose X is realized as a closed subvariety in $\mathbf{P}^n$. Using method of problem 6 show that for any invertible $\mathcal{O}$-module L on X the $\mathcal{O}$-module $L(k)$ is very ample for large k.

Definition. Consider an abelian group A and denote by $F(A)$ group of all integer valued functions on A (the group structure is given by addition of functions).

For any $a \in A$ we define operators $T_a, \Delta_a : F(A) \to F(A)$ by
$T_a(f)(x) = f(x-a), \Delta_a(f)(x) = f(x) - f(x-a)$
(these are called the translation and the difference operator).

For every $l \in \mathbf{Z}$ we define the subgroup $Pol^{\leq l} \subset F(A)$ of **polynomial functions of degree** $\leq l$ as follows:

If $l < 0$ we set $Pol^{\leq l} = 0$
If $l \geq 0$ we define the subgroups $Pol^{\leq l}$ inductively
$Pol^{\leq l}$ consists of all functions $f \in F(A)$ such that for any $a \in A$ we have $\Delta_a(f) \in Pol^{leql-1}$.

8. Let X be a projective variety, $A = Pic(X)$ its Picard group. For every coherent $\mathcal{O}_X$-module we define a function P_F on the group A by $P_F(L) := \chi(L \otimes F)$.

Show that this is a polynomial function on A of degree $\leq \dim supp(F)$.

Hint. Use the result of problem 7.

9. Consider the projection $p : X = \mathbf{P}^n \times Y \to Y$ and a coherent $\mathcal{O}_X$-module F as in problem 6.

(i) Show that there exists a resolution $\mathcal{Q}^\cdot$ of F, $0 \to \mathbf{Q}_0 \to Q_1 \to Q_n \to 0$, that consists of coherent p_*-acyclic modules. Show that this resolution can be constructed in semi-functorial way.

(ii) Assume in addition that the module F is flat over Y. Show that we can choose resolution $\mathcal{Q}^\cdot$ such that all modules Q_i are flat over Y. Show that in this case the complex of $\mathcal{O}_Y$-modules $K^\cdot = p_*(\mathcal{Q})$ is a complex of locally free $\mathcal{O}_Y$-modules that allows to compute derived functors $\mathbf{R}^i p_*$ for F and for all of its base changes.

Hint. Show that for every $k \geq 0$ the $\mathcal{O}$-module F can be naturally imbedded in a module $\mathbf{R}_k$ that is a direct sum of several copies of the module $F(k)$ so that locally this imbedding is an isomorphism with a direct summand. Then use results of problem 5 in assignment 14.

This gives another, more constructive, proof of Grothendieck's result.