Problem assignment 17.

Algebraic Geometry and Commutative Algebra 2.

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In this assignment we will use the following important result

Serre's Duality Theorem. Let X be a smooth projective variety of dimension n, ω_X the canonical bundle of X - the top exterior power of the cotangent bundle Ω^1_X.

To any invertible $\mathcal{O}$-module L we assign "adjoint" $\mathcal{O}$-module $\hat{L}$ defined by $\hat{L} = \mathcal{H}om(L, \omega_X) = L^* \otimes \omega_X$.

Then for any i we have a perfect pairing $H^i(X, L) \times H^{n-i}(X, \hat{L}) \to k$. In particular, $\dim H^i(X, L) = \dim H^{n-i}(X, \hat{L})$.

1. Prove the following Riemann-Roch theorem for surfaces

Theorem. Let S be a smooth projective surface. Then for any $L \in Pic(S)$ we have $\chi(L) = -1/2B(L, \hat{L}) + \chi(\mathcal{O}_X)$.

Hint. Using the fact that $\chi(L)$ is a quadratic function on $Pic(S)$ and B is the corresponding bilinear form deduce that both sides of the equality differ by a linear function. Using identity $\chi(L) = \chi(\hat{L})$ show that they differ by a constant.

Let S be a connected smooth projective surface. Fix a very ample bundle $H \in Pic(S)$ and introduce a quasidegree d on the group $A = Pic(S)$ by $d(a) = B(H, a)$. Our aim is to prove the Hodge Index Theorem that asserts that the form B is negative on the orthogonal complement to H in $Pic(S)$.

For $L \in Pic(S)$ we set $h_i(L) := \dim H^i(S, L)$.

Theorem. (i) $B(H, H) > 0$

(ii) If $d(a) = 0$, then $B(a, a) \leq 0$.

2. Show that if an invertible $\mathcal{O}$-module L has negative quasidegree, i.e. $d(L) < 0$, then $h_0(L) = 0$.

3. Show that there exists a function $\Phi(n)$ on $\mathbf{Z}$ such that for any invertible $\mathcal{O}$-module L we have $h_0(L) \leq \Phi(d(L))$.

Hint. Choose a section of H such that the variety E of its zeroes is a smooth curve (maybe not connected). Show that $h_0(L) \leq h_0(L(-E)) + d(L) + C$. Then use induction.

In fact we can choose $\Phi(n)$ to be quadratic in n but we do not need this result.

4. Show that there exists a function $\Psi(n)$ on $\mathbf{Z}$ such that for any invertible $\mathcal{O}$-module L we have $h_2(L) \leq \Psi(d(L))$.

Hint. Problem 3 and Serre's duality.

5. Let $a \in Pic(S)$ be an element with $d(a) = 0$. Consider elements $a_n = na \in Pic(S)$. Show that $\chi(a_n)$ is bounded above. Show that $\chi(a) = 1/2\ n^2 B(a, a)+$ linear function in n. Deduce Hodge Index Theorem.

Let C be a smooth curve of genus g. We will use the theory of surfaces to get some non-trivial bounds for C.

Consider the surface $S = C \times C$. In the group $Div(S)$ consider elements v-vertical fiber and h-horizontal fiber. We consider them as elements of the group $A = Pic(S)$. Let us denote by $A^{\perp}$ the orthogonal complement to v and h in A. Every element $x \in A$ can be written as $mv + nh + x'$ where $x' \in A^{\perp}$. Clearly $B(mv + nh + x', pv + qh + y') = mq + np + B(x', y')$.

We would like to use the result that the form B is negative on the group $A^{\perp}$ - this follows from Hodge index Theorem (and from problems in assignment 16).

6. Let Δ be the diagonal divisor. Show that $B(\Delta, \Delta) = 2 - 2g$.

Fix a morphism $F : S \to S$ and denote by d its degree. The morphism F defines a morphism $F_S : S \to S$ by formula $F_S(x, y) = (F(x), y)$.

This defines an endomorphism f of the group $A = Pic(S)$ by $f(L) = F_S^*(L)$.

7. (i) Show that $B(f(a), f(b)) \equiv d\ f(a, b)$
(ii) Show that $B(f(a), v) = d\ B(a, v)$ and $B(f(a), h) = B(a, h)$.

8. Prove the following Weil bound on the intersection index of Δ and $F_S^*(\Delta)$:
$B(F_S^*(\Delta), \Delta) = d + 1 + e$, where $|e| \leq 2g\sqrt{d}$
Hint. Let $a = \Delta,\ b = f(\Delta) \in S$. Show that
$B(a, a) = 2 - 2g,\ B(a', a') = -2g$
$B(b, b) = d(2 - 2g),\ B(b', b') = -2dg$
$B(a, b) = d + 1 + B(a', b')$
After this use the fact that the form B is negative on the subgroup $A^{\perp}$.

Ample and very ample line bundles.

Definition. Let X be a projective variety, L – invertible $\mathcal{O}_X$-module. We say that L is **ample** if for any coherent $\mathcal{O}_X$-module F there exists a number $n_F > 0$ such that for any $n > n_F$ the $\mathcal{O}$-module $F \otimes L^{\otimes n}$ is generated by its global sections.

9. (i) Show that an invertible $\mathcal{O}$-module is ample iff some positive power $L^{\otimes n}$ is very ample.

(ii) Show that (i) holds for arbitrary algebraic variety X if we modify the notion of the very ample bundle appropriately.

Namely an invertible $\mathcal{O}_X$-module L is called **very ample** if some collection of its sections defines a morphism $\nu : X \to \mathbf{P}^N$ that gives an isomorphism of X with some locally closed subvariety $Y \subset \mathbf{P}^N$.