Problem assignment 9.

Algebraic Geometry and Commutative Algebra II

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Let A be a commutative algebra. We would like to consider collections C of A-modules and morphisms between them of some type. We say that some property P of this collection is **localizable** if for any such collection C the following conditions are equivalent:

(i) S has property P

(ii) For any multiplicatively closed subset $S \subset A$ the localized collection $S^{-1}C$ of $S^{-1}A$-modules satisfies P

(iii) For nay maximal ideal $\mathfrak{m} \subset A$ the localized collection $C_{\mathfrak{m}}$ satisfies P

1. Find out which of the following properties are localizable

(i) Morphism $\nu : M \to N$ is monomorphism ?; epimorphism ?; isomorphism ?

(ii) Complex C is exact at the place 0

(iii) Two submodules $L.N \subset M$ have zero intersection ?; generate M ?

(iv) An A-algebra B is integral over A

(*) (v) An A-module M is finitely generated

2. Let B be an A-algebra. Show that an element $b \in B$ is integral over A iff this is true locally on $SpecA$.

3. Let A be a domain, K its field of fractions. For any maximal ideal $\mathfrak{m} \subset A$ we consider the localized algebra $A_{\mathfrak{m}}$ as a subalgebra in K.

(i) Show that A is equal to the intersection of subalgebras $A_{\mathfrak{m}}$ corresponding to all maximal ideals.

(ii) Show that A is integrally closed iff all algebras $A_{\mathfrak{m}}$ are integrally closed.

4. Let A be a domain, K its field of fractions and L/K be a finite field extension. We would like to show that under some conditions the algebra $B = Int(A; L)$ is finite over A.

Let us assume that the algebra A is Noetherian and integrally closed.

(i) Show that if L/K is a separable extension then the algebra B is finite over A.

(ii) Let A be an algebra over a field k of characteristic p and assume that A is finite over the subalgebra $A' = kA^p$ (for example this holds if A is finitely generated k-algebra).

Show that in this case B is finite over A.

5. Let A be a unique factorization domain. Show that it is integrally closed.

Definition. An algebraic variety X is called **normal** if every local ring of X is an integral and integrally closed domain.

6. Write this definition more geometrically, in terms of open affine coverings of X.

7. Show that in algebra geometric case for any domain A and any finite extension L/K of its field of fractions K the integral closure $B = Int(A, L)$ is finite over A.

Using this fact show that any irreducible algebraic variety X has normalization $\hat{X}$. Show that the natural morphism $p : \hat{X} \to X$ is finite and birational.

Show that for any normal irreducible variety Z any dominant morphism $Z \to X$ can be uniquely lifted to a morphism $Z \to \hat{X}$.