Problem assignment 2.

Algebraic Theory of D-modules.

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Filtrations.

Let us fix a field k.

F1. Definition. Let V be a vector space.

An (increasing) **filtration** Φ on V is a collection of subspaces $V_k = \Phi_k V$ for all $k \in \mathbf{Z}$ satisfying the following properties

(i) V_k is an increasing sequence, i.e. $V_i \subset V_k$ for $i \leq k$.

(ii) **Separation.** $V_k = 0$ for $k \ll 0$

(iii) **Exhaustion.** $V = \bigcup V_k$.

A morphism of filtered vector spaces from V, Φ to W, Ψ is a linear operator $\nu : V \to W$ such that $\nu(V_k) \subset W_k$ for all k.

For a filtered vector space V, Φ we denote by $gr^{\Phi}(V)$ the associated **graded** vector space defined as follows:

$gr^{\Phi}(V) := \oplus_k gr_k^{\Phi}(V)$, where $gr_k^{\Phi}(V) := V_k / V_{k-1}$

1. (i) Let V, Ω be a filtered vector space. Let L be a subspace of V and $N = V/L$ be the quotient space, so we have a short exact sequence $0 \to L \to V \to N \to 0$.

Show that there exists unique pair of filtrations Φ on L and Ψ on N such that morphisms $L \to V$ and $V \to N$ are morphisms of filtered spaces and the corresponding morphisms

$0 \to gr^{\Phi} L \to gr^{\Omega} V \to gr^{\Psi} N \to 0$ form a short exact sequence.

Describe these filtrations explicitly. Characterize them by universal properties.

These filtrations are called **induced filtrations** on subspace and quotient space.

(ii) Let $\nu : V \to W$ be a morphism of filtered spaces and $\lambda = gr(\nu) : gr(V) \to gr(W)$ the induced morphism of associated graded spaces. Show that if λ is mono then ν is mono; if λ is epi, then ν is epi; if λ is an isomorphism, then ν is an isomorphism.

F2. Definition. (i) Let A be an associative algebra with 1. A filtration A_k on A is called an **algebra filtration** if it satisfies

(i) $A_k A_l \subset A_{k+l}$

(ii) $1 \in A_0$

We usually also assume the following condition

(iii) $A_i = 0$ for $i < 0$

A filtered algebra is an algebra equipped with an algebra filtration.

(ii) Let A be a filtered algebra and M an A-module. A **module filtration** on M is a filtration M_i which satisfies $A_k M_l \subset M_{k+l}$

An A-module M equipped with a module filtration we will call a **filtered A-module**.

2. (i) Show that if A, F is a filtered algebra then the space $\Sigma = gr^F A$ has a natural structure of a **graded** associative algebra with 1.

(ii) Show that if M is a filtered A-module then the space $M_{\Sigma} := grM$ has a natural structure of a **graded** Σ-module.

F3. We will fix a filtered algebra A, F. We assume that it satisfies the following conditions

(i) The associated graded algebra Σ is commutative. In particular A_0 is a commutative algebra.

(ii) Σ is a finitely generated k-algebra.

(iii) Usually we also assume that Σ is generated by $\Sigma_0 = A_0$ and Σ_1

Definition. Let A, F be a filtered algebra and M, Φ be a filtered A-module.

We say that Φ is a **good** filtration if the associated graded Σ-module M_{Σ} is finitely generated.

3. Let A, F be a filtered algebra as above. In particular the associated graded algebra Σ is Noetherian.

(i) Let Φ be a good filtration on an A-module M. Let L be a submodule of M and $N := M/L$. Show that the induced filtrations Φ_L on L and Φ_N on N are good.

(ii) Show that an A-module M admits a good filtration if and only if it is finitely generated.
(iii) Show that the algebra A is Noetherian.

4. Let Φ, Ψ be two good filtrations on an A-module M.

(i) Show that filtrations Φ, Ψ are comparable.
Namely, there exists N such that for any k, $\Phi_k \subset \Psi_{k+N}$ and $\Psi_k \subset \Phi_{k+N}$.

(ii) Show that there exists a sequence of filtrations Φ^l such that for any l filtrations Φ^{l+1} and Φ^l are neighbors, for $l << 0$ we have $\Phi^l = \Psi$ and for $l >> 0$ we have $\Phi^l = \phi[l]$.

Show that the associate graded Σ-modules $M_{\Phi,\Sigma}$ and $M_{\Psi,\Sigma}$ are Jordan-Hoelder equivalent as **ungraded** Σ-modules.

F4. Definition. From now on let us assume that the field k is algebraically closed. Consider the algebraic variety $\Theta = Specm(\Sigma) = Mor_{k-alg}(\Sigma, k)$.

For any finitely generated A-module M define its support to be a subset $S_M \subset \Theta$, closed in Zariski topology. that is the support of Σ-module M_Σ.

Let us define the **functional dimension** $d(M)$ of a finitely generated A-module M by $d(M) := \dim S_M$

5. (i) Show that if $0 \to L \to M \to N \to 0$ is an exact sequence of A-modules, then $S_M = S_L \bigcup S_N$
(ii) Show that if a finitely generated A-module M is a sum of submodules L_κ, then $S_M = \bigcup S_{L_\kappa}$.
(*) (iii) Show that the functional dimension $d(M)$ does not depend on the choice of filtration F on the algebra A.

6. Suppose A-module M is generated by one element m. Pick an element $a \in A_d$ and denote by σ its image in Σ_d.

Show that σ vanishes on the support S_M of the module M iff the function $\mathrm{def}_{a,m}(i) := \deg(a^i m) - d \cdot i$ tends to $-\infty$ when i goes to infinity, i.e the degree of $a^i m$ is much smaller than expected degree.

7. Let us assume that $charK = 0$. Prove the Kashiwara's lemma
Lemma Let D be a k-algebra generated by two elements t, ∂ such that $[\partial, t] = 1$.
Let M be an arbitrary D-module. We set $L := \mathrm{Ker}(t, M)$; $V_i := \partial^i(L)$ for $i = 0, 1, ...$

(i) Show that the product $d = t\partial$ acts on V_i as a scalar $-1 - i$.
(ii) Show that the spaces V_i are linearly independent, operator $t : V_i \to V_{i-1}$ is an isomorphism for $i > 0$ and operator $\partial : V_i \to V_{i+1}$ is an isomorphism for $i \geq 0$.
(iii) Show that the space $M' = \oplus V_i$ is a D-submodule of M, the operator $t : M' \to M'$ is an epimorphism and the operator t does not have kernel on the quotient model $C = M/M'$.

8. Let A be a k-algebra. Suppose A contains a subalgebra isomorphic to D in problem 7. Also assume that the operator $ad(t) : A \to A$ is locally nilpotent. Denote by B the centralizer of t in A, i.e. $B = \mathrm{Ker}\ ad(t)$.
(i) Show that for any A-module M the space $L = \mathrm{Ker}(t, M)$ is a B-module.
(ii) Let $\mathcal{M}'(a)$ be the full subcotegory of the category $\mathcal{M}(A)$ of A-modules that consists of modules M on which the operator t is locally nilpotent.
Show that the functor $M \mapsto \mathrm{Ker}(t, M)$ gives an equivalence of category $\mathcal{M}'(A)$ with the category $\mathcal{M}(B)$ of B-modules.
Describe the inverse functor.