Problem assignment 3.

Algebra B3 – Commutative Algebra

Joseph Bernstein November 15, 2012.

Definition. Let C be a category. An object $F \in Ob(C)$ is called **a final object** if for any object $X \in Ob(C)$ the set $Mor_C(X, F)$ consists of exactly one element.

Similarly one defines the notion of **an initial object**.

1. (i) Show that if there exists a final object F in C then it is defined uniquely up to unique isomorphism.

(ii) Let C be a category and R an object of C. We define a new category C_R - category of objects over R - as follows

Object of C_R is a pair (X, p), where $X \in Ob(C)$ and $p : X \to R$.

Morphism $\nu : (X, p) \to (Y, q)$ in the category C_R is a morphisms $\nu : X \to Y$ in C that makes the diagram commute, i.e. $q \cdot \nu = p$.

Show that the category C_R always has a final object.

(iii) Let R, S be objects of C. Define in the natural way the category C_{RS}.

If this category has a final object (Z, p_R, p_S) we call the object $Z \in Ob(C)$ the product of objects R, S and the morphisms p_R, p_S projections.

As follows from (i) this product is defined uniquely up to canonical isomorphism. Check for yourself that this definition coincides with the one discussed in class.

(iv) Define the notion of sum of objects in a category.

2. In any category C define the notions of "monomorphism", "epimorphism", "isomorphisms". Give example when "monomorphism + epimorphism does not imply isomorphism.

[P] 3. Let $\mathcal{A}$ be an abelian category. Consider a complex $0 \to L \to M \to N \to 0$ of objects in $\mathcal{A}$.

(i) Show that this complex is left exact iff for any object X the complex of abelian groups $0 \to Hom(X, L) \to Hom(X, M) \to Hom(X, N) \to 0$ is left exact.

(ii) Show that this complex is right exact iff for any object Z the complex of abelian groups $0 \to Hom(N, Z) \to Hom(M, Z) \to Hom(L, Z) \to 0$ is left exact.

4. Fix an algebra A and consider the category $\mathcal{M}(A)$ of A-modules.

(i) Show that this category has arbitrary (e.g. infinite) sums.

[P] (ii) Define the notion of the tensor product of A-modules. Show that for any A-module N the functor $T_N : \mathcal{M}(A) \to \mathcal{M}(A)$ defined by $T_N(M) := M \otimes_A N$ is right exact and preserves direct sums (note that it does not always preserve direct products).

Using this fact describe a procedure for computing the tensor product.

(ii) Compute $A/I \otimes_A A/J$

[P] 5. (i) Fix a morphism of algebras $\nu : A \to B$. Consider an A-linear functor $F : \mathcal{M}(A) \to \mathcal{M}(B)$ (define precisely this notion).

Suppose that the functor F satisfies the following conditions:

(R1) F is right exact

(R2) F preserves arbitrary direct sums.

Show that then F is isomorphic to a functor $T_N : M \mapsto N \otimes M$ for some B-module N.

Show that one can not omit the condition R2.

(ii) Let S be a multiplicatively closed subset of A. Show that there exists a canonical functorial isomorphism $S^{-1}M \simeq S^{-1}A \otimes_A M$ for $M \in \mathcal{M}(A)$.

Definition. An A-module R is called **flat** if the functor $T_R : M \to R \otimes_A M$ is exact.

[P] 6. (i) Show that any projective A-module is flat. Show that the tensor product of flat modules is flat.

(ii) Let S be a multiplicatively closed subset of A. Consider the functors $Res : \mathcal{M}(S^{-1}A) \to \mathcal{M}(A)$ and $Ext : \mathcal{M}(A) \to \mathcal{M}(S^{-1}A)$.
Show that they send flat modules into flat.

Definition. Fix an algebra A and for every prime ideal $\mathfrak{p}$ consider the localization functor $\mathcal{M}(A) \to \mathcal{M}(A_\mathfrak{p})$.
We say that some property $\mathcal{P}$ of modules is local if $\mathcal{P}$ holds for an A-module M iff it holds for localizations $M_\mathfrak{p}$ for all prime ideals of A. Similarly for properties of morphisms of modules.
This is a very convenient condition since it allows to reduce many questions to the case of local algebras.

[P] 7. (i) Show that the following properties are local
(a) $M = 0$
(b) Morphism $\nu : M \to N$ is zero, or mono, or epi, or isomorphism.
(c) A complex is exact at place i.
(d) Module M is flat

(ii) Show that if a module M is projective then all its localizations are projective. Is the converse true ?
The same question for property to be finitely generated.

Definition. An A-module M is called **Noetherian** if any A-submodule $L \subset M$ is finitely generated.
An algebra A is called Noetherian if any finitely generated A-module M is Noetherian.

[P] 8. (i) Show that an A-module M is Noetherian iff it satisfies the following condition
(*) Any increasing chain of submodules $L_1 \subset L_2 \subset ... \subset M$ is stable, i.e. modules L_i coincide for large i-s.

(ii) Let M be an extension of L and N. Show that M is Noetherian iff L and N are Noetherian.

(iii) Show that an algebra A is Noetherian iff A is Noetherian as A-module.

9. (i) Let $A = k[t_1, t_2, ...]$ be the algebra of polynomials in infinite number of generators. Show that A is not Noetherian

(ii) Let $A = k[x, y]$. Consider a subalgebra $B = k + xA \subset A$. Show that the algebra B is not Noetherian (and hence not finitely generated as k-algebra).

[P] 10. Let M be a Noetherian A-module. Show that the $A[t]$-module $M[t]$ is also Noetherian.