Automorphic Forms - Home Assignment 2

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Question 1

In this question, we will explicitly perform the computation of $\operatorname{Pic} E$ for an elliptic curve E. Let $\Lambda \subseteq \mathbb{C}$ be a lattice, $E = \mathbb{C}/\Lambda$. As shown in class, we are interested in (not necessarily linear!) entire holomorphic nowhere-zero functions c_ω such that

$$c_{\omega+\omega'} = c_\omega \cdot T_\omega c_{\omega'},$$

where $T_\omega(f)(z) = f(z+\omega)$ is translation, modulo functions of the form

$$\frac{T_\omega u}{u},$$

where u is holomorphic.

(a) Show that since c_ω is never zero, it is the exponent of some entire holomorphic function ℓ_ω. That is, $c_\omega = \exp(\ell_\omega)$. Conclude that we are interested in entire holomorphic functions ℓ_ω such that:

$$\ell_{\omega+\omega'} \equiv \ell_\omega + T_\omega \ell_{\omega'} \pmod{2\pi i},$$

modulo additions of $2\pi i$ to any individual function ℓ_ω and modulo all functions of the form

$$T_\omega u - u,$$

for some entire u.

(b) Let $\Delta_\omega f = f(z+\omega) - f(z)$. Show that we may assume that $\ell_0 = 0$, and that $\ell_{\omega_1} = \ell_1$, $\ell_{\omega_2} = \ell_2$ determine ℓ_ω for any $\omega \in \Lambda$, where ω_1, ω_2 are a basis for Λ. Conclude that we are looking for pairs (ℓ_1, ℓ_2) such that

$$\Delta_{\omega_1} \ell_2 \equiv \Delta_{\omega_2} \ell_1 \pmod{2\pi i},$$

modulo pairs of the form $(2\pi i\mathbb{Z}, 2\pi i\mathbb{Z})$, and of the form

$$(\Delta_{\omega_1} u, \Delta_{\omega_2} u),$$

for some entire u.

(c) Let Ω be the skew-symmetric bilinear form on Λ satisfying

$$\Omega(\omega_1, \omega_2) = \frac{1}{2\pi i}\left(\Delta_{\omega_1}\ell_2 - \Delta_{\omega_2}\ell_1\right).$$

Show that Ω is a well-defined skew-symmetric integer-valued bilinear form on Λ. Extend Ω to $\mathbb{C}$ and show that if $\Omega = 2\operatorname{Im} H$, where H is some Hermitian form on $\mathbb{C}$, then $(\hat{\ell}_1, \hat{\ell}_2) = (H(\omega_1, z), H(\omega_2, z))$ is a cocycle (satisfies the conditions we imposed on ℓ_1, ℓ_2). Conclude that we may assume that $\Omega = 0$.

(d) By choosing an appropriate u, show that we may assume that $\ell_1 = 0$. We are thus reduced to a single function ℓ such that

$$\Delta_{\omega_1} \ell = 0,$$

(that is, ℓ is periodic), modulo $2\pi i \mathbb{Z}$ and all functions $\Delta_{w_2} u$ such that u is entire and

$$\Delta_{\omega_1} u \equiv 0 \pmod{2\pi i}.$$

(e) Reduce to considering all ω_1-periodic functions, modulo the constants $2\pi i$, $2\pi i \frac{\omega_2}{\omega_1}$ and modulo all functions of the form $\Delta_{w_2} u$ where u is ω_1-periodic. By taking the Fourier series of ℓ, show that we are left (after choosing u appropriately) with all *constant* functions ℓ modulo $\frac{2\pi i}{\omega_1} \Lambda$.

(f) Conclude that $\operatorname{Pic} E = \mathbb{Z} \times E$, where the projection onto $\mathbb{Z}$ is given by considering the integer-valued skew-symmetric bilinear form Ω of section (c), and the projection onto E is given by considering $\frac{1}{2\pi i} \omega_1 \ell$, where ℓ is the constant at the end of section (e).

Question 2

In this section we explicitly compute the four classical theta functions. We consider four cocycles defined by (we let $k, l \in \{0, 1\}$):

$$\begin{aligned} c_{\omega_1} &= 1, \\ c_{\omega_2} &= (-1)^l \exp(2\pi i \frac{z}{\omega_1} + k\pi i \frac{\omega_2}{\omega_1}). \end{aligned}$$

Note that these four cocycles are of degree one, shifted by all four half-periods. Define the corresponding line bundle $\mathcal{L}(\omega_1, \omega_2, k, l)$ by:

$$\Gamma(U, \mathcal{L}(\omega_1, \omega_2, k, l)) = \left\{ f : p^{-1}(U) \to \mathbb{C} \mid f(z) = \Pi_\omega f(z) := c_\omega f(z + \omega) \right\},$$

where $p : \mathbb{C} \to E$ is the projection. We also let $L(\omega_1, \omega_2, k, l) = \Gamma(E, \mathcal{L}(\omega_1, \omega_2, k, l))$ be the global sections of these line bundles. We will obtain the classical theta functions as global sections of these bundles. Please keep in mind that there are many notations for the classical theta functions, so it is easy to confuse different notations. Our notation of the theta functions will be that which is most convenient for our current purposes, but it will be non-standard.

(a) Verify that the functions $(c_{\omega_1}, c_{\omega_2})$ defined above are indeed cocycles and that they are non-trivial (hint: use the computation of $\operatorname{Pic} E$).

(b) Show that $L(\omega_1, \omega_2, k, l)$ are one-dimensional complex linear spaces, spanned by the functions

$$\Theta_{00}(\omega_1, \omega_2; z) = \sum_{n=-\infty}^{\infty} \exp\left(2\pi i \binom{n}{2} \frac{\omega_2}{\omega_1}\right) \exp\left(2\pi i n \frac{z}{\omega_1}\right) \in L(\omega_1, \omega_2, 0, 0)$$

$$\Theta_{01}(\omega_1, \omega_2; z) = \sum_{n=-\infty}^{\infty} (-1)^n \exp\left(2\pi i \binom{n}{2} \frac{\omega_2}{\omega_1}\right) \exp\left(2\pi i n \frac{z}{\omega_1}\right) \in L(\omega_1, \omega_2, 0, 1)$$

$$\Theta_{10}(\omega_1, \omega_2; z) = \sum_{n=-\infty}^{\infty} \exp\left(\pi i n^2 \frac{\omega_2}{\omega_1}\right) \exp\left(2\pi i n \frac{z}{\omega_1}\right) \in L(\omega_1, \omega_2, 1, 0)$$

$$\Theta_{11}(\omega_1, \omega_2; z) = \sum_{n=-\infty}^{\infty} (-1)^n \exp\left(\pi i n^2 \frac{\omega_2}{\omega_1}\right) \exp\left(2\pi i n \frac{z}{\omega_1}\right) \in L(\omega_1, \omega_2, 1, 1).$$

(c) Prove the identities:

$$\begin{aligned}
\Theta_{kl}(\omega_1,\omega_2;z+\omega_1) &= \Theta_{kl}(\omega_1,\omega_2;z)\\
\Theta_{kl}(\omega_1,\omega_2;z+\omega_2) &= (-1)^l \exp\left(-2\pi i\frac{z}{\omega_1} - k\pi i\frac{\omega_2}{\omega_1}\right)\Theta_{kl}(\omega_1,\omega_2;z)\\
\Theta_{kl}(\lambda\omega_1,\lambda\omega_2;\lambda z) &= \Theta_{kl}(\omega_1,\omega_2;z)\\
\Theta_{kl}(\omega_1,\omega_2+\omega_1;z) &= \Theta_{k(k+l)}(\omega_1,\omega_2;z),
\end{aligned}$$

where $k+l$ is the XOR of k and l.

(d) Show that

$$\begin{aligned}
&\Theta_{00}(\omega_2,-\omega_1;z)\exp\left(-\pi i\frac{z^2}{\omega_1\omega_2} - \pi i\frac{z}{\omega_2}\right) &&\in L(\omega_1,\omega_2,1,1)\\
&\Theta_{11}(\omega_2,-\omega_1;z)\exp\left(-\pi i\frac{z^2}{\omega_1\omega_2} + \pi i\frac{z}{\omega_1}\right) &&\in L(\omega_1,\omega_2,0,0)\\
&\Theta_{01}(\omega_2,-\omega_1;z)\exp\left(-\pi i\frac{z^2}{\omega_1\omega_2} + \pi i\frac{z}{\omega_1} - \pi i\frac{z}{\omega_2}\right) &&\in L(\omega_1,\omega_2,0,1)\\
&\Theta_{10}(\omega_2,-\omega_1;z)\exp\left(-\pi i\frac{z^2}{\omega_1\omega_2}\right) &&\in L(\omega_1,\omega_2,1,0),
\end{aligned}$$

and conclude that there is some holomorphic $\alpha_{kl}(\omega_1,\omega_2)$ such that

$$\Theta_{kl}(\omega_2,-\omega_1;z) = \Theta_{(1-l)(1-k)}(\omega_1,\omega_2;z)\alpha_{kl}(\omega_1,\omega_2)\exp\left(\pi i\frac{z^2}{\omega_1\omega_2} - l\pi i\frac{z}{\omega_1} + (1-k)\pi i\frac{z}{\omega_2}\right).$$

Finally, show that:

$$\Theta_{kl}(1,-1/\tau;z/\tau) = \Theta_{(1-l)(1-k)}(1,\tau;z)\alpha_{kl}(1,\tau)\exp\left(\pi i\frac{z^2}{\tau} - l\pi iz + (1-k)\pi i\frac{z}{\tau}\right).$$

(e) Suppose that $f \in L(1,\tau,k,l)$. Show that for real x,y, the function

$$|f(x+y\tau)|^2\exp\left(-2\pi\,\mathrm{Im}(\tau)(y^2-(1-k)y)\right)$$

is invariant under $x \mapsto x+1$, $y \mapsto y+1$. Deduce that

$$H_{kl,\tau}(f) = \int_0^1\int_0^1 |f(x+y\tau)|^2\exp\left(-2\pi\,\mathrm{Im}(\tau)(y^2-(1-k)y)\right)\mathrm{d}x\mathrm{d}y$$

is a well-defined Hermitian form on $L(1,\tau,k,l)$.

(f) Compute that

$$H_{(1-l)(1-k)\tau}\left(\Theta_{(1-l)(1-k)}(1,\tau;z)\right) = \exp\left(l\frac{\pi\,\mathrm{Im}(\tau)}{2}\right)\frac{1}{\sqrt{2\,\mathrm{Im}(\tau)}},$$

and

$$\begin{aligned}
H_{(1-l)(1-k)\tau}&\left(\Theta_{kl}(1,-1/\tau;z/\tau)\exp\left(-\pi i\frac{z^2}{\tau} + l\pi iz - (1-k)\pi i\frac{z}{\tau}\right)\right) =\\
&= \exp\left((1-k)\frac{\pi\,\mathrm{Im}(\tau)}{2|\tau|^2}\right)\frac{|\tau|}{\sqrt{2\,\mathrm{Im}(\tau)}},
\end{aligned}$$

and conclude that

$$|\alpha_{kl}(1,\tau)|^2 = |\tau| \exp\left(\frac{\pi \operatorname{Im}(\tau)}{2}(\frac{1-k}{|\tau|^2} - l)\right).$$

Using the holomorphicity of $\alpha_{kl}(1,\tau)$, show that there are unitary constants C_{kl} such that

$$\Theta_{kl}(1,-1/\tau;z/\tau) = C_{kl}\cdot \tau^{1/2}\exp\left(\frac{\pi i}{4}(l\tau + \frac{1-k}{\tau})\right)\times$$
$$\times \exp\left(\pi i \frac{z^2}{\tau} - l\pi i z + (1-k)\pi i \frac{z}{\tau}\right)\Theta_{(1-l)(1-k)}(1,\tau;z).$$

Note that the values of these constants can be obtained more directly by applying the Poisson summation formula to the infinite sums defining $\Theta_{kl}(1,\tau;z)$.

(g) Show that if we let $\theta_{kl}(\tau) = \Theta_{kl}(1,\tau;0)$, then (note that $\theta_{01}(\tau)\equiv 0$):

$$\begin{aligned}
\theta_{00}(-1/\tau) &= C_{00}\cdot \tau^{1/2}\exp\left(\frac{\pi i}{4\tau}\right)\theta_{11}(\tau)\\
\theta_{11}(-1/\tau) &= C_{11}\cdot \tau^{1/2}\exp\left(\frac{\pi i\tau}{4}\right)\theta_{00}(\tau)\\
\theta_{10}(-1/\tau) &= C_{10}\cdot \tau^{1/2}\theta_{10}(\tau),
\end{aligned}$$

and:

$$\begin{aligned}
\theta_{00}(\tau+1) &= \theta_{00}(\tau)\\
\theta_{11}(\tau+1) &= \theta_{10}(\tau)\\
\theta_{10}(\tau+1) &= \theta_{11}(\tau).
\end{aligned}$$

Furthermore, show that $C_{00}C_{11} = C_{10}^2 = 1$.

(h) Combine everything to obtain that $\theta_{10}(2\tau)$ is a modular form of weight $\frac{1}{2}$ for the modular group $\Gamma_0(4) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0 \pmod 4\right\}$, possibly with some character (this is the theta function whose modularity we have already seen via the Poisson summation formula in the previous home assignment!).