Automorphic Forms - Home Assignment 3

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Question 1

In this question we will investigate the structures of various congruence groups:

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\}$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$

A subgroup $\Gamma \subseteq \Gamma(1) = SL_2(\mathbb{Z})$ will be called a *congruence subgroup* if it contains one of the subgroups $\Gamma(N) \subseteq \Gamma(1)$. Our goal is to compute the indices of the above congruence groups in $\Gamma(1) = SL_2(\mathbb{Z})$.

(a) Consider the morphism:

$$SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z}).$$

Show that it is an epimorphism and that its kernel is $\Gamma(N)$.

(b) Show that there is an isomorphism $SL_2(\mathbb{Z}/N\mathbb{Z}) \cong SL_2(\mathbb{Z}/p_1^{k_1}\mathbb{Z}) \times \cdots \times SL_2(\mathbb{Z}/p_m^{k_m}\mathbb{Z})$, where $N = p_1^{k_1} \cdots p_m^{k_m}$ is the prime decomposition of N.

(c) Prove that the kernel of the projection $SL_2(\mathbb{Z}/p^k\mathbb{Z}) \to SL_2(\mathbb{Z}/p\mathbb{Z})$ has size $p^{3(k-1)}$ for all $k \geq 1$.

(d) Compute that $|SL_2(\mathbb{Z}/p\mathbb{Z})| = p(p^2-1)$. Conclude that

$$[SL_2(\mathbb{Z}) : \Gamma(N)] = N^3 \prod_{p|N} \frac{p^2-1}{p^2}.$$

(e) Similarly, let $A_0(\mathbb{Z}/N\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in SL_2(\mathbb{Z}/N\mathbb{Z}) \right\}$. Prove that the kernel of the projection $A_0(\mathbb{Z}/p^k\mathbb{Z}) \to A_0(\mathbb{Z}/p\mathbb{Z})$ has size $p^{2(k-1)}$. Compute that $|A_0(\mathbb{Z}/p\mathbb{Z})| = p(p-1)$. Conclude that

$$[\Gamma_0(N) : \Gamma(N)] = N^2 \prod_{p|N} \frac{p-1}{p},$$

$$[SL_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p|N} \frac{p+1}{p}.$$

(f) Similarly, let $A_1(\mathbb{Z}/N\mathbb{Z}) = \left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z}/N\mathbb{Z})\right\}$. Prove that $|A_1(\mathbb{Z}/N\mathbb{Z})| = N$. Conclude that

$$[\Gamma_1(N) : \Gamma(N)] = N,$$
$$[SL_2(\mathbb{Z}) : \Gamma_1(N)] = N^2 \prod_{p|N} \frac{p^2-1}{p^2}.$$

Question 2

In this question we will investigate the cusp structures of various congruence groups: we denote the upper half plane with infinity and rational points by $\bar{H} = H \cup \mathbb{Q} \cup \{\infty\}$. Note that $\bar{H}$ is closed under the action of $\Gamma(1) = SL_2(\mathbb{Z})$, and that the points of $\mathbb{Q} \cup \{\infty\}$ all lie in a single orbit of $\Gamma(1)$.

(a) Show that the compactification $\hat{Y_1}$ of $Y_1 = \Gamma(1)\backslash H$ is isomorphic to $\Gamma(1)\backslash\bar{H}$. Use this to show that for any subgroup $\Gamma \subseteq \Gamma(1)$ of finite index, $\hat{Y} = \Gamma\backslash\bar{H}$ is the compactification of $Y = \Gamma\backslash H$ defined in class. Because of this, we will refer to points of $\mathbb{Q} \cup \{\infty\}$ as *cusps*, and we will say that two of them are equivalent for Γ if they lie in the same Γ-orbit. Our goal is now to classify inequivalent cusps.

(b) Let $a \in \mathbb{Q}\cup\{\infty\}$ be a cusp and Γ be a congruence subgroup. Let $\Gamma_a = \{\gamma \in \Gamma \mid \gamma a = a\}$, and let $\Gamma(1)_\infty = \{\gamma \in SL_2(\mathbb{Z}) \mid \gamma\infty = \infty\} \cong \{\pm 1\} \times \mathbb{Z}$. Finally, let $\gamma_a \in SL_2(\mathbb{Z})$ be some element such that $\gamma_a\infty = a$. Show that $\gamma_a^{-1}\Gamma_a\gamma_a \subseteq \Gamma(1)_\infty$ is a subgroup of finite index, generated by an element of the form $\begin{pmatrix} 1 & m_a \\ 0 & 1 \end{pmatrix}$ for some m_a, and possibly ± 1. Show that $m_a | N$, where $\Gamma(N) \subseteq \Gamma$, i.e. N is the level of Γ. We refer to m_a as the *multiplicity* of the cusp a.

(c) Use the fact that a complete list of coset representatives for $\Gamma(N)\backslash SL_2(\mathbb{Z})$ is given by matrices in $SL_2(\mathbb{Z}/N\mathbb{Z})$, to show that two cusps $\frac{a}{c}$ and $\frac{a'}{c'}$ (where $(a,c) = (a',c') = 1$) are equivalent for $\Gamma(N)$ iff $(a,c) \equiv \pm(a',c') \pmod{(N\mathbb{Z}, N\mathbb{Z})}$.

Furthermore, show that the multiplicites of each of these cusps are equal to N. Note that we let $\frac{1}{0} = \infty$ in the above list.

(d) Find an example for a cusp with multiplicity $m_a > 1$ for the group $\Gamma_0(N)$. What is the multiplicity m_∞ of the cusp at ∞ for this group?

Question 3

In this question we will represent the various families of congruence subgroups as moduli spaces of lattices with level structure.

(a) Show that:

$$\Gamma(N)\backslash H = \{\text{lattices } \Lambda \text{ with isomrphism } \alpha : (\mathbb{Z}/N\mathbb{Z})^2 \xrightarrow{\sim} \Lambda/N\Lambda\}/\mathbb{C}^\times.$$

(b) Show that:

$$\Gamma_0(N)\backslash H = \{\text{lattices } \Lambda \text{ with distinguished line } L \subseteq \Lambda/N\Lambda\}/\mathbb{C}^\times.$$

(c) Show that:

$$\Gamma_1(N)\backslash H = \{\text{lattices } \Lambda \text{ with monomorphism } \alpha : \mathbb{Z}/N\mathbb{Z} \to \Lambda/N\Lambda\}/\mathbb{C}^\times.$$

Question 4

In this question we will prove the multiplication relations for Hecke operators for congruence subgroups. Recall that if $f : \tilde{M}(N) \to \mathbb{C}$ is a homogenous function of degree k form the set of lattices Λ with level structures $\alpha : (\mathbb{Z}/N\mathbb{Z})^2 \xrightarrow{\sim} \Lambda/N\Lambda$, then (for $(m, N) = 1$):

$$T_m(f)(\Lambda) = m^{k-1} \sum_{\substack{\Lambda' \subseteq \Lambda \\ |\Lambda/\Lambda'|=m, \alpha'=\alpha}} f(\Lambda').$$

Also recall that homogenous of degree k means that $f(\lambda\Lambda) = \lambda^{-k} f(\Lambda)$.

(a) Use the fact that if $(m, n) = 1$, then any Abelian group Λ/Λ'' of size mn has a unique exact sequence

$$0 \to \Lambda'/\Lambda'' \to \Lambda/\Lambda'' \to \Lambda/\Lambda' \to 0,$$

where Λ'/Λ'' is of size m and Λ/Λ' is of size n to show that

$$T_m T_n = T_{mn}.$$

(b) Show that if Λ/Λ'' is a finite quotient of two lattices, then it is isomorphic to $(\mathbb{Z}/a\mathbb{Z}) \times (\mathbb{Z}/b\mathbb{Z})$ for some a, b, and further that if a, b have a common denominator c, then Λ'' is of the form $c\hat{\Lambda}''$ for some sublattice $\hat{\Lambda}''$ of Λ.

You may use the *Cartan decomposition*, which means that for any 2×2 matrix g with integer coefficients,

$$g = k_1 \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} k_2,$$

where $k_1, k_2 \in SL_2(\mathbb{Z})$.

(c) Suppose that Λ/Λ'' is a finite quotient of two lattices, of size $p^{\nu+1}$. Then show that either it is of the form $\mathbb{Z}/p^{\nu+1}\mathbb{Z}$, or $\Lambda'' = p\hat{\Lambda}''$ for some $\hat{\Lambda}''$. In the first case, prove that there is a unique lattice $\Lambda'' \subseteq \Lambda' \subseteq \Lambda$ such that Λ'/Λ'' is of size p. In the second case, show that there are $p + 1$ such Λ'. Conclude that

$$\sum_{\substack{\Lambda'' \subseteq \Lambda \\ |\Lambda/\Lambda''|=p^{\nu+1}}} f(\Lambda'') = \sum_{\substack{\Lambda'' \subseteq \Lambda' \subseteq \Lambda \\ |\Lambda/\Lambda'|=p^{\nu}, |\Lambda'/\Lambda''|=p}} f(\Lambda'') - p \sum_{\substack{\hat{\Lambda}'' \subseteq \Lambda \\ |\Lambda/\hat{\Lambda}''|=p^{\nu-1}}} f(p\hat{\Lambda}'').$$

(d) Prove that for any $(p, N) = 1$,

$$T_{p^{\nu+1}} = T_p T_{p^{\nu}} - p^{k-1} T_{p^{\nu-1}},$$

and conclude that for any $(m, N) = 1$, $(n, N) = 1$,

$$T_m T_n = \sum_{r|(m,n)} r^{k-1} T_{\frac{mn}{r^2}}.$$

(e) Prove the above relations for Hecke operators for modular forms of $\Gamma_0(N)$ and $\Gamma_1(N)$ as well.