Automorphic Forms - Home Assignment 4

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Question 1

Our goal in this question is to identify (as much as one can) the functions appearing in the Fourier series of a Maass form.

(a) Let f be a Maass form of Laplace eigenvalue μ (that is, f is invariant under $\Gamma(1) = SL_2(\mathbb{Z})$, has moderate growth, and satisfies $\Delta f = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f = \mu f$). Show that

$$f(x+iy) = \sum_{n\in\mathbb{Z}} W_n(y)\exp(2\pi i n x),$$

for some functions of moderate growth $W_n(y)$.

(b) Let $n \neq 0$. Show that if we let $W_n(y) = y^{1/2}F_n(y)$ then $F_n(y)$ satisfies the differential equation

$$(-4\pi^2y^2n^2 - \mu - \frac{1}{4})F_n(y) + yF_n'(y) + y^2F_n''(y) = 0.$$

Let $F_n(y) = K_n(ny)$. So, $K_n(y)$ is of moderate growth and satisfies

$$(-4\pi^2y^2 - \mu - \frac{1}{4})K_n(y) + yK_n'(y) + y^2K_n''(y) = 0.$$

This differential equation was studied by Whittaker, and its solutions are called *Whittaker functions.*

(c) Still assuming that $n \neq 0$, show that for y sufficiently large, the equation has two linearly independent solutions, one growing as $O(y^{-1/2}\exp(2\pi y))$, and the other growing as $O(y^{-1/2}\exp(-2\pi y))$. Also show that near $y = 0$, both solutions exhibit at most polynomial growth.

(d) Use the fact that f has moderate growth to deduce that $K_n(y) = a_nK(y)$ for some coefficients a_n, and some function $K(y)$.

(e) Let $n = 0$. Then show that $W_0(y)$ satisfies

$$y^2\frac{\partial}{\partial y}W_0(y) = \mu W_0(y).$$

Solve this to see that if we let $\mu = \lambda^2 - \frac{1}{4}$, then

$$W_0(y) = y^{1/2}(Ay^{\lambda} + By^{-\lambda}).$$

Note that in this case, both solutions of the differential equation have moderate growth.

(f) Conclude that

$$f(x+iy) = y^{1/2}(Ay^{\lambda} + By^{-\lambda}) + \sum_{n\neq 0} a_n y^{1/2}K(ny)\exp(2\pi i n x).$$

Question 2

Our goal in this question is to compute the L-function of a Maass form and its functional equation.

Let

$$f(x+iy) = y^{1/2}(Ay^{\lambda} + By^{-\lambda}) + \sum_{n\neq 0} a_n y^{1/2} K(ny)\exp(2\pi inx)$$

be a Maass cusp form ($A = B = 0$), of Laplacian eigenvalue μ, as above. Let $L(f,s) = \sum_{n>0} a_n n^{-s}$.

(a) Suppose that f is even, that is, $f(x+iy) = f(-x+iy)$. Show that $a_n = a_{-n}$, and compute that

$$M(y^{-1/2}f(iy);s) = \int_0^{\infty} y^{-1/2}f(iy)y^s \frac{\mathrm{d}y}{y} = 2G(s-\frac{1}{2})L(f,s),$$

where $G(s) = M(y^{1/2}K(y);s)$ is the Mellin transform of $y^{1/2}K(y)$. Conclude that

$$G(s-\frac{1}{2})L(f,s) = G(\frac{1}{2}-s)L(f,1-s).$$

(b) Suppose that f is odd, that is, $f(x+iy) = -f(-x+iy)$. Show that $a_n = a_{-n}$, and compute that

$$M(y^{-1/2}f(iy);s) = 0.$$

(c) To solve this (we must have some sort of Mellin transform in order to prove the functional equation on $L(f,s)$), we would like to consider $\frac{\partial}{\partial x}f$ instead of f - which will be an even function, and have a good Mellin transform. However, this is not entirely correct; it is conceptually better if we were to consider $y^2\frac{\partial}{\partial x}f$ instead, as this seems to be in order with the invariant metric on the upper half plane we have seen in class. However, an even better operator to consider would be $R\cdot f = y(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})f$. This operator is called the *Maass raising operator*.

The justification for this name comes form the fact that $\frac{1}{y}R\cdot f$ transforms like a modular form of weight 2 (that is, an application of R raises the weight from 0 to 2). Prove this.

(d) Compute that

$$M(y^{-1/2}(Rf)(iy);s) = \int_0^{\infty} y^{-1/2}(Rf)(iy)y^s\frac{\mathrm{d}y}{y} = 4\pi iG(s+\frac{1}{2})L(f,s),$$

where $G(s) = M(y^{1/2}K(y);s)$ is as above. Conclude that

$$G(s+\frac{1}{2})L(f,s) = -G(\frac{3}{2}-s)L(f,1-s).$$

Note the extra minus sign!

Question 3

Our computation above left us with the unknown function

$$G(s) = M(y^{1/2}K(y);s) = \int_0^{\infty} y^{1/2}K(y)y^s\frac{\mathrm{d}y}{y}.$$

We expect that this function can be expressed via the Gamma function. In this question, we will compute it. Our strategy will be rather indirect, and in a sense we will be *solving* the differential equation defining $K(y)$.

(a) Using the growth properties of $K(y)$ shown above, prove that the integral $G(s)$ converges absolutely for $\operatorname{Re} s > 0$. Use the differential equation for $K(y)$ to show that for $\operatorname{Re} s$ sufficiently large,

$$4\pi^2 G(s+2) + \mu G(s) - s(s+1)G(s) = 0.$$

(b) Conclude that if we let $(s+\alpha)(s+\beta) = s(s+1) - \mu$, then

$$\frac{G(y)}{\pi^{-s}\Gamma(\frac{s+\alpha}{2})\Gamma(\frac{s+\beta}{2})}$$

is a periodic function in s, of period 2.

(c) You may assume the well known analytic fact that within the domain of absolute convergence, the Mellin transform of a function is bounded in vertical strips. Also note that $\Gamma(s) = M(\exp(-y); s)$ is the Mellin transform of $\exp(-y)$. Use this to show that the function

$$\frac{G(y)}{\pi^{-s}\Gamma(\frac{s+\alpha}{2})\Gamma(\frac{s+\beta}{2})}$$

is periodic of period 2, and holomorphic and bounded in some right half plane $\operatorname{Re} s > M$. Conclude that it is constant. Note that this gives us an expression for $K(y)$ as an integral (via the inverse Mellin transform).

(d) Finally, recall that we let $\mu = \lambda^2 - \frac{1}{4}$, $L(f,s) = \sum_{n>0} a_n n^{-s}$, to show that for all even Maass cusp forms f,

$$\pi^{\frac{1}{2}-s}\Gamma(\frac{s+\lambda}{2})\Gamma(\frac{s-\lambda}{2})L(f,s) = \pi^{s-\frac{1}{2}}\Gamma(\frac{1-s+\lambda}{2})\Gamma(\frac{1-s-\lambda}{2})L(f,1-s),$$

and for all odd Maass cusp forms,

$$\pi^{-\frac{1}{2}-s}\Gamma(\frac{1+s+\lambda}{2})\Gamma(\frac{1+s-\lambda}{2})L(f,s) = -\pi^{-\frac{3}{2}+s}\Gamma(\frac{2-s+\lambda}{2})\Gamma(\frac{2-s-\lambda}{2})L(f,1-s).$$