Automorphic Forms - Home Assignment 7

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Our primary goal in this exercise is to explicitly compute the L-function $L(\chi, s)$ for the global field $\mathbb{Q}$, and to find its functional equation.

Question 1

We first need to do some (rather trivial) Fourier transforms on p-adic numbers. Recall that the *p-adic exponent* $e_p : \mathbb{Q}_p^+ \to \mathbb{C}^\times$ is given by:

$$e_p(x) = \exp(-2\pi i \{x\}),$$

where $\{x\}$ is the fractional part of x, that is, a rational number $\{x\} \in \mathbb{Q}$ such that $x - \{x\} \in \mathbb{Z}_p$. Let $f \in S(\mathbb{Q}_p)$ be any Schwartz function (locally constant and compactly supported function). Define the Fourier transform $F\{f\}$ of f to be the integral

$$F\{f\}(y) = \int_{\mathbb{Q}_p} f(x) e_p(-xy) \mathrm{d}x,$$

where $\mathrm{d}x$ is the standard Haar measure on $\mathbb{Q}_p$, that is, the unique measure satisfying $\mu(1 + p^n\mathbb{Z}_p) = p^{-n}$ (in particular, the set $\mathbb{Z}_p$ has volume 1).

We also denote (for a set $A \subseteq \mathbb{Q}_p$):

$$\mathbf{1}_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that

$$F\{\mathbf{1}_{\mathbb{Z}_p}\} = \mathbf{1}_{\mathbb{Z}_p}.$$

(b) Show that

$$F\{\mathbf{1}_{p^n\mathbb{Z}_p}\} = p^{-n} \mathbf{1}_{p^{-n}\mathbb{Z}_p}.$$

(c) Show that

$$F\{\mathbf{1}_{a+p^n\mathbb{Z}_p}\}(y) = e_p(-ay) p^{-n} \mathbf{1}_{p^{-n}\mathbb{Z}_p}.$$

Question 2

Now, for the interesting part. Let $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ be a smooth (that is, locally constant) multiplicative character. Recall that

$$\mathcal{E}_{\chi,s}(\phi) = Z_p(\chi, \phi, s) = \int_{\mathbb{Q}_p} \phi(x)\chi(x)|x|^s \mathrm{d}^\times x.$$

(a) Recall the definition of $\gamma_p(\chi, s)$, given by $F(\mathcal{E}_{\chi,s}) = \gamma_p(\chi, s)\mathcal{E}_{\chi^{-1},1-s}$. By applying both sides to the test function $\phi = \mathbf{1}_{\mathbb{Z}_p}$, show that when χ is unramified, we have

$$\gamma_p(\chi, s) = \frac{1 - \chi(p)^{-1}p^{s-1}}{1 - \chi(p)p^{-s}}.$$

(b) When χ is ramified, define its *conductor* N to be the least integer such that $\chi|_{1+p^N\mathbb{Z}_p} = 1$. That is, we have that χ is constant and equal to $\chi(1) = 1$ on the neighborhood $1 + p^N\mathbb{Z}_p$ of 1, and N is the least such number. Using the test function $\phi = \mathbf{1}_{1+p^N\mathbb{Z}_p}$, show that for χ ramified,

$$Z_p(\chi^{-1}, \phi, 1-s) = \frac{p}{p-1}p^{-N},$$

$$Z_p(\chi, F\{\phi\}, s) = \frac{p}{p-1}p^{-N}\frac{p^{Ns}}{\chi(p^N)}\frac{1}{p^N}\sum_{\substack{j=0\\(j,p)=1}}^{p^N-1} e^{\frac{2\pi i j}{p^N}}\chi(j).$$

Conclude that,

$$\gamma_p(\chi, s) = \frac{p^{Ns}}{\chi(p^N)}\frac{1}{p^N}\sum_{\substack{j=0\\(j,p)=1}}^{p^N-1} e^{\frac{2\pi i j}{p^N}}\chi(j).$$

(c) Recall that we defined the local L-function to be $L_p(\chi, s) = (1-\chi(p)p^{-s})^{-1}$ when χ is unramified, and that we defined $L_p(\chi, s) = 1$ when χ is ramified. Finally, recall that we had:

$$\frac{Z_p(\chi, F\{\phi\}, s)}{L_p(\chi, s)} = \varepsilon_p(\chi, s)\frac{Z_p(\chi^{-1}, \phi, 1-s)}{L_p(\chi^{-1}, 1-s)}.$$

Deduce that

$$\varepsilon_p(\chi, s) = \begin{cases} 1 & \text{if } \chi \text{ is unramified,} \\ \frac{p^{Ns}}{\chi(p^N)}\frac{1}{p^N}\sum_{\substack{j=0\\(j,p)=1}}^{p^N-1} e^{\frac{2\pi i j}{p^N}}\chi(j) & \text{if } \chi \text{ is ramified.} \end{cases}$$

Question 3

Our goal now is to compute the L-function at ∞. So, let $\chi : \mathbb{R}^\times \to \mathbb{C}^\times$ be some smooth multiplicative character. Let $\chi(r) = |r|^\nu \operatorname{sign}(r)^\delta$, where $\delta = 0, 1$. When $\delta = 0$, we say that χ is *unramified*, and if $\delta = 1$, we say that χ is *ramified*.

Define

$$\mathcal{E}_{\chi,s}(\phi) = Z_\infty(\chi, \phi, s) = \int_{\mathbb{R}} \phi(x)\chi(x)|x|^s \mathrm{d}^\times x,$$

where $\mathrm{d}^\times x = \frac{\mathrm{d}x}{|x|}$.

(a) Show that for any Schwartz function $\phi \in S(\mathbb{R})$, the function $Z_\infty(\chi, \phi, s)$ converges absolutely for $\mathrm{Re}(s + \nu) > 0$.

(b) We would like to show that $Z_\infty(\chi, \phi, s)$ has meromorphic continuation in s, and can only have specific poles. Show that if $\phi = \phi_0 + \phi_1$, where ϕ_1 is supported away from 0 (so that $\phi_1(x) = 0$ for $|x|$ sufficiently small), then the poles and meromorphic continuation properties of $Z_\infty(\chi, \phi, s)$ depend only on ϕ_0.

(c) Suppose that ϕ is analytic in some neighborhood of 0 (that is, it has a Taylor series at 0 converging to itself in some neighborhood).

Show that if $\delta = 0$, then $Z_\infty(\chi, \phi, s)$ can only have poles at

$$s + \nu = 0, -2, -4, -6, \ldots,$$

with residues

$$\phi(0), \frac{\phi^{(2)}(0)}{2!}, \frac{\phi^{(4)}(0)}{4!}, \frac{\phi^{(6)}(0)}{6!}, \ldots$$

respectively.

Show that if $\delta = 1$, then $Z_\infty(\chi, \phi, s)$ can only have poles at

$$s + \nu = -1, -3, -5, \ldots,$$

with residues

$$\phi'(0), \frac{\phi^{(3)}(0)}{3!}, \frac{\phi^{(5)}(0)}{5!}, \ldots$$

respectively.

(d) Use the fact that anayltic around 0 functions are dense among Schwartz functions to show that the above pole structure applies to any Schwartz function.

(e) We now know that if we choose some Schwartz function ϕ such that none of its derivatives at 0 are zero (and if χ is unramified, we also need that $\phi(0) \neq 0$), then the quotient

$$\frac{Z_\infty(\chi, \psi, s)}{Z_\infty(\chi, \phi, s)}$$

will be holomorphic for any other Schwartz function ψ. This gives us the greatest common divisor of all local zeta functions, but it is only defined up to multiplication by a holomorphic function. So, we must simply make a choice. A particularly convenient choice is $\phi(x) = \exp(-\pi x^2)x^\delta$.

Show that for this choice of ϕ, we have

$$L_\infty(\chi, s) = Z_\infty(\chi, \phi, s) = \pi^{-(s+\nu+\delta)/2}\Gamma\left(\frac{s + \nu + \delta}{2}\right).$$

Question 4

The last ingredient we need is the computation of the local root number at infinity $\varepsilon_\infty(\chi, s)$, defined by

$$\frac{Z_\infty(\chi, F\{\phi\}, s)}{L_\infty(\chi, s)} = \varepsilon_\infty(\chi, s)\frac{Z_\infty(\chi^{-1}, \phi, 1 - s)}{L_\infty(\chi^{-1}, 1 - s)}$$

Recall that the Fourier transform at infinity is given by

$$F\{\phi\}(y) = \int_{\mathbb{R}} \phi(x)e^{-2\pi ixy}\mathrm{d}x.$$

(a) Suppose that χ is unramified (so $\delta = 0$). Compute that $F\{\phi\} = \phi$, and deduce that in this case,

$$\gamma(\chi, s) = \frac{Z_\infty(\chi, F\{\phi\}, s)}{Z_\infty(\chi^{-1}, \phi, 1 - s)} = \frac{L_\infty(\chi, s)}{L_\infty(\chi^{-1}, 1 - s)} = \frac{\pi^{-(s+\nu)/2}\Gamma\left(\frac{s+\nu}{2}\right)}{\pi^{-(1-s-\nu)/2}\Gamma\left(\frac{1-s-\nu}{2}\right)}.$$

Colnclude that if χ is unramified, then $\varepsilon_\infty(\chi, s) = 1$.

(b) Suppose that χ is ramified (so $\delta = 1$). Compute that $F\{\phi\} = \frac{1}{i}\phi$, and deduce that in this case,

$$\gamma(\chi, s) = \frac{Z_\infty(\chi, F\{\phi\}, s)}{Z_\infty(\chi^{-1}, \phi, 1 - s)} = \frac{1}{i}\frac{L_\infty(\chi, s)}{L_\infty(\chi^{-1}, 1 - s)} = \frac{1}{i}\frac{\pi^{-(s+\nu+1)/2}\Gamma\left(\frac{s+\nu+1}{2}\right)}{\pi^{-(2-s-\nu)/2}\Gamma\left(\frac{2-s-\nu}{2}\right)}.$$

Conclude that if χ is ramified, then $\varepsilon_\infty(\chi, s) = i^{-1}$.

Question 5

Let us wrap everything up. Let $\chi : \mathbb{A}_{\mathbb{Q}}^{\times}/\mathbb{Q}^{\times} \to \mathbb{C}^{\times}$ be a grossencharacter. Let $\chi = \prod \chi_v$ be its decomposition over primes. Then we have that

$$L(\chi, s) = \prod L_v(\chi_v, s)$$

satisfies

$$L(\chi^{-1}, 1-s) = \varepsilon(\chi, s) L(\chi, s) = \left(\prod \varepsilon_v(\chi_v, s)\right) L(\chi, s).$$

So, we wish to compute the product of the local root numbers $\prod \varepsilon_v(\chi_v, s)$, which were calculated above. We denote $\chi_\infty = |\cdot|^\nu \operatorname{sign}(\cdot)^\delta$.

Let

$$S = \{\text{finite primes } p \mid \chi \text{ is ramified at } p\}.$$

(a) For any Dirichlet character $\rho : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}^{\times}$, define the *Gauss sum*,

$$G(\rho) = \frac{1}{N} \sum_{\substack{j=0 \\ (j,N)=1}}^{N-1} e^{\frac{2\pi i j}{N}} \rho(j).$$

Show that if N, N' are coprime, and $\rho' : \mathbb{Z}/N'\mathbb{Z} \to \mathbb{C}^{\times}$ is also a Dirichlet character, then

$$G(\rho\rho') = \rho(N')\rho'(N) G(\rho) G(\rho').$$

(b) We denote the conductor of χ_p by N_p, and we let $N = \prod_{p \in S} p^{N_p}$. Also denote

$$\tilde{\chi}(j) = \begin{cases} \prod_{p \in S} \chi_p^{-1}(j) & \text{if } (j, N) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Show that $\tilde{\chi}$ is a primitive Dirichlet character modulo N.
Also show that:

$$\prod_{p \in S} \left(\frac{1}{\chi_p(p^{N_p})} \sum_{\substack{j=0 \\ (j,p)=1}}^{p^{N_p}-1} e^{\frac{2\pi i j}{p^{N_p}}} \chi_p(j) \right) = \frac{1}{\prod_{p \in S} \chi_p(N)} G(\tilde{\chi}^{-1}).$$

(c) Show that $\chi(N) = \chi_\infty(N) \prod_{p \in S} \chi_p(N)$. Conclude that since χ is a grossencharacter, then this product is equal to 1.
Also show that if $p \notin S$, then $\chi_p(p) = \chi_\infty^{-1} \tilde{\chi}(p) = \tilde{\chi}(p) p^{-\nu}$.

(d) Finally, summarize everything and show that if we let

$$L(\chi, s) = \pi^{-(s+\nu+\delta)/2} \Gamma\left(\frac{s+\nu+\delta}{2}\right) \prod_{p \notin S} (1 - \tilde{\chi}(p) p^{-(s+\nu)})^{-1}$$

(where $\chi_\infty = |\cdot|^\nu \operatorname{sign}(\cdot)^\delta$, as above) then

$$L(\chi^{-1}, 1-s) = \frac{1}{i^\delta} N^{(s+\nu)} G(\tilde{\chi}^{-1}) L(\chi, s).$$

(e) In particular, let $\chi = 1$ be the trivial character, with $N = 1$, $\nu = \delta = 0$, $S = \emptyset$. Show that in this case, if we let

$$L(1, s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\prod_p (1 - p^{-s})^{-1} = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

(where $\zeta(s)$ is the Riemann zeta function) then

$$L(1, s) = L(1, 1 - s).$$

(Note that we have been implicitly assuming that $N \neq 1$ when multiplying the Gauss sums above. Verify that everything is still valid when $N = 1$.)