Automorphic Forms - Home Assignment 8

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Our goal in this exercise will be to illustrate the Langlands correspondence by considering a specific example.

Question 1

Let $K = \mathbb{Q}$, and L be the splitting field of the polynomial $x^3 + x + 1 = 0$. We will denote its roots by $\alpha, \beta, \gamma \in L$. We will be working with this example throughout this assignment. First of all, let us begin by doing some Galois theory!

(a) Show that $\beta^2 + \alpha\beta + (\alpha^2 + 1) = 0$, and $\gamma = -\alpha - \beta$. From now on, we will consider L to be the explicit splitting field $\mathbb{Q}[\alpha, \beta, \gamma]/ < \alpha^3 + \alpha + 1, \beta^2 + \alpha\beta + (\alpha^2 + 1), \alpha + \beta + \gamma >$.

(b) Show that $L = \mathbb{Q}[\alpha, \beta] = \mathbb{Q}[\alpha, \sqrt{\Delta}]$, where $\Delta = -31$ is the discriminant of the polynomial $x^3 + x + 1$. Conclude that L/K is Galois with Galois group $G = \mathrm{Gal}(L/K) = S_3$.

Question 2

Our current goal is to realize the behavior of the Artin L-function corresponding to a specific non-trivial Galois representation ρ.

Let L, K as above, and let $\rho : G = \mathrm{Gal}(L/K) = S_3$ be the unique 2-dimensional irreducible representation of S_3. Explicitly, it is given by considering S_3 as the group generated by a $120°$-rotation and a reflection:

$$(1 \quad 2) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$(1 \quad 2 \quad 3) \mapsto \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}.$$

Let us determine the behavior of the corresponding Artin L-function.

(a) Case **??**: let p be a prime such that the polynomial $x^3 + x + 1$ is irreducible in $\mathbb{Z}/p\mathbb{Z}$. Show that the Frobenius element Fr_p must act as a cycle on the roots of $x^3 + x + 1$ in L, and deduce that the local L-function is:

$$L_p(\rho, s) = \det(1 - p^{-s}\rho(Fr_p))^{-1} = (1 + p^{-s} + p^{-2s})^{-1}.$$

(b) Case **??**: let p be a prime such that the polynomial $x^3 + x + 1$ is completely reducible in $\mathbb{Z}/p\mathbb{Z}$ (that is, has 3 *distinct* roots). Show that the Frobenius element Fr_p must not affect any of the roots of $x^3 + x + 1$ in L. Deduce that the local L-function is:

$$L_p(\rho, s) = \det(1 - p^{-s}\rho(Fr_p))^{-1} = (1 - 2p^{-s} + p^{-2s})^{-1}.$$

(c) Case **??**: let p be a prime such that the polynomial $x^3 + x + 1$ has exactly one root in $\mathbb{Z}/p\mathbb{Z}$. Show that the Frobenius element Fr_p must act as a transposition on two of the roots of $x^3 + x + 1$ in L, and deduce that the local L-function is:

$$L_p(\rho, s) = \det(1 - p^{-s}\rho(Fr_p))^{-1} = (1 - p^{-2s})^{-1}.$$

(d) Case **??**: let $p = 31$. Show that the polynomial $x^3 + x + 1$ has decomposition $(x - 3)(x + 17)^2$ in $\mathbb{Z}/31\mathbb{Z}$. Show that the inertia subgroup I in this case consists of a single transposition, and conclude that ρ is ramified at 31. Show that in this case, the Frobenius element Fr_{31} acts trivially on L. Deduce that the local L-function is:

$$L_{31}(\rho, s) = \det(1 - 31^{-s}\rho(Fr_{31})\big|_{V^I})^{-1} = (1 - 31^{-s})^{-1},$$

where $V \cong \mathbb{C}^2$ is the space on which ρ acts, and V^I is the subspace invariant under the action of the inertia subgroup.

Question 3

Our current goal is to compute by hand the first few terms of the Artin L-function corresponding to the above Galois representation ρ, and to show that the above list of cases is exhaustive.

(a) Show that case **??** occurs if and only if $\Delta = -31$ has no square root in $\mathbb{Z}/p\mathbb{Z}$.

(b) Conclude by quadratic reciprocity that case **??** occurs if and only if p has no square root modulo 31. Equivalently, show that case **??** occurs iff

$$p \equiv 3, 6, 11, 12, 13, 15, 17, 21, 22, 23, 24, 26, 27, 29, 30 \pmod{31}.$$

(c) Show that at least one of the cases **??**, **??**, **??** and **??** must occur. That is, ρ is ramified only at $p = 31$.

Remark 1. The following is a computer-generated list of the first few primes, and the cases to which they belong:

prime	case	$L_p(\rho, s)$
$p = 2$	??	$(1 + 2^{-s} + 2^{-2s})^{-1}$
$p = 3$	??	$(1 - 3^{-2s})^{-1}$
$p = 5$	??	$(1 + 5^{-s} + 5^{-2s})^{-1}$
$p = 7$	??	$(1 + 7^{-s} + 7^{-2s})^{-1}$
$p = 11$	??	$(1 - 11^{-2s})^{-1}$
$p = 13$	??	$(1 - 13^{-2s})^{-1}$
$p = 17$	??	$(1 - 17^{-2s})^{-1}$
$p = 19$	??	$(1 + 19^{-s} + 19^{-2s})^{-1}$
$p = 23$	??	$(1 - 23^{-2s})^{-1}$
$p = 29$	??	$(1 - 29^{-2s})^{-1}$
$p = 31$	??	$(1 - 31^{-s})^{-1}$
$p = 37$	??	$(1 - 37^{-2s})^{-1}$
$p = 41$	??	$(1 + 41^{-s} + 41^{-2s})^{-1}$
$p = 43$	??	$(1 - 43^{-2s})^{-1}$
$p = 47$	??	$(1 - 2 \cdot 47^{-s} + 47^{-2s})^{-1}$
$p = 53$	??	$(1 - 53^{-2s})^{-1}$
$p = 59$	??	$(1 + 59^{-s} + 59^{-2s})^{-1}$
$p = 61$	??	$(1 - 61^{-2s})^{-1}$
$p = 67$	??	$(1 - 2 \cdot 67^{-s} + 67^{-2s})^{-1}$
$p = 71$	??	$(1 + 71^{-s} + 71^{-2s})^{-1}$

Question 4

Now that we have an L-function, what can we say about the corresponding automorphic representation? We will assume throughout this question that there exists some irreducible automorphic representation (π, V) of GL_2 such that we have an equality of L-functions (up to terms at ∞):

$$L(\pi, s) = \prod_p L_p(\pi, s) = L(\rho, s) = \prod_p L_p(\rho, s).$$

Note that we are assuming that the above equation holds exactly, with the product over all finite primes, including 31.

(a) Prove (via some very general anayltic nonsense) that $L_p(\rho, s) = L_p(\pi, s)$ for all finite primes p.

(b) Let $z : \mathbb{A}_{\mathbb{Q}}^\times \to Z(GL_2(\mathbb{A}_{\mathbb{Q}}))$ be defined by $z(r) = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$. Use Schur's lemma to show that there is some grossencharacter $\omega_\pi : \mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times \to \mathbb{C}^\times$ such that

$$\pi(z(r)) \cdot f = \omega_\pi(r) f$$

for all $r \in \mathbb{A}_{\mathbb{Q}}^\times$ and $f \in V$. This ω_π is called the *central character* of π.

(c) Show that for all finite primes $p \neq 31$, we have that the local character $\omega_{\pi,p} : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ is unramified and satisfies

$$\omega_{\pi,p}(p) = \left(\frac{p}{31}\right) = \begin{cases} 1 & \text{if } p \text{ has a square root modulo 31,} \\ -1 & \text{if } p \text{ has no square root modulo 31,} \end{cases}$$

which is also known as the *Legendre symbol.*

(d) Use the fact that ω_π is a grossencharacter to deduce that:

$$\omega_{\pi,31}(a \cdot 31^n) = \left(\frac{a}{31}\right),$$

for all $n \in \mathbb{Z}$, $a \in \mathbb{Z}_{31}$, and that

$$\omega_{\pi,\infty}(x) = \operatorname{sign} x$$

for all $x \in \mathbb{R}^\times$.

Remark 2. Note that in fact, in addition to determining the central character of π, we can also determine the *level* of the classical modular form to which it corresponds. Indeed, we have seen that π is only ramified at 31 and ∞ (the fact that it is ramified at ∞ follows from the fact that its central character is ramified at ∞).

Furthermore, we note that the L-function at 31 is not equal to 1, it is not quadratic, *and* π_{31} has a ramified central character. It is possible to see that this only happens if π_{31} is a *principal series representation*, induced from an unramified character χ_1 and a ramified character χ_2 such that $\chi_1(31) = 1$ is the coefficient of -31^{-s} in the L-function. Since $\chi_1\chi_2 = \omega_{\pi,31}$, we also know χ_2. The explicit definition of the principal series representation can then be played to show that π_{31} always has a $K_{0,1}(31)$-fixed vector, where

$$K_{0,1}(31) = \left\{ A \in GL_2(\mathbb{Z}_p) \mid A \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \pmod{31} \right\}.$$

Thus, we see that the automorphic representation π contains an automorphic form f invariant to

$$\left\{ A \in GL_2(\mathbb{A}_{\text{fin}}) \mid A_{31} \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \pmod{31} \right\}.$$

In particular, f corresponds to a classical form of level 31, and character $\chi(a) = \left(\frac{a}{31}\right)$.