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Characterizing the value functions of polynomial games

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ABSTRACT

We provide a characterization of the set of real-valued functions that can be the value function of some polynomial game. Specifically, we prove that a function $u : \mathbf{R} \to \mathbf{R}$ is the value function of some polynomial game if and only if u is a continuous piecewise rational function.

1. Introduction

A *polynomial game* is a finite strategic-form game whose payoffs are polynomials in a real-valued parameter Z. The value of the two-player zero-sum polynomial game depends on the value of Z, and therefore it is a function $u : \mathbf{R} \to \mathbf{R}$. Since the value of a two-player zero-sum strategic-form game with finitely many strategies is a solution of a set of linear inequalities (see, e.g., [6]), it follows that the value function u is a continuous and piecewise rational function; that is, one can divide the real line $\mathbf{R}$ into finitely many intervals such that the function u is a rational function on each piece. In this note we show that the converse also holds: every continuous and piecewise rational function is the value function of some polynomial game.

The reader should not confuse polynomial games as defined above with the class of games bearing the same name and studied in [2], which are two-player zero-sum strategic-form games in which the action set of each player is $[0, 1]$ and the payoff function is polynomial in x and y, where $x \in [0, 1]$ is the mixed action chosen by Player 1 and $y \in [0, 1]$ is the mixed action chosen by Player 2.

Characterizing the collection of value functions of a given model has several reasons. First, the richness of the set of value functions indicates the complexity of the model, and allows us to compare models that seem unrelated. Second, each restriction that the value function must satisfy arises from some aspect(s) in the model, hence increases our understanding of the model. Third, sometimes we are given the value for some parameters, and we would like to estimate the value for other parameters. Once we identify the set of possible value functions of the model, we know how many data points we need to estimate the value function for new parameters. Fourth, sometimes we are given the value function for some parameters, yet we do not completely know the underlying model. The characterization of the set of value functions may allow us to rule out possible models or increase our confidence in a prospective model. The characterization of the set of value functions of Markov decision processes appears in [3]. Characterizations of the set of equilibrium payoffs of nonzero-sum games are provided in [4,5], and [7].

2. The model and the main result

Definition 2.1. A two-player zero-sum polynomial game is a tuple $\Gamma = (A, B, G)$, where A is the finite strategy set of Player 1, B is the finite strategy set of Player 2, and $G = ((G_{a,b})_{a \in A, b \in B})$ is the payoff matrix of size $|A| \times |B|$, whose entries are polynomials in Z.

For every $z \in \mathbf{R}$, denote by $\Gamma(z) = (A, B, G(z))$ the two-player zero-sum strategic-form game where the sets of strategies of the two players are A and B respectively, and the payoffs are $G(z) = (G_{a,b}(z))_{a \in A, b \in B}$, that is, the payoffs in Γ evaluated at $Z = z$. Denote by $u(z) := \text{val}(\Gamma(z))$ the value in mixed strategies of the game $\Gamma(z)$. The function $u : \mathbf{R} \to \mathbf{R}$ is the *value function* of the polynomial game Γ. Denote by $\mathcal{V}$ the set of all real-valued functions that are value functions of some polynomial game Γ.

Definition 2.2. A function $u : \mathbf{R} \to \mathbf{R}$ is *piecewise rational* if there are a natural number $K \in \mathbf{N}$, a sequence $-\infty = h_1 < h_2 < \cdots < h_K = \infty$, and rational functions $(\frac{Q_k}{R_k})_{k=1}^{K-1}$ such that

$$u = \sum_{k=1}^{K-1} \mathbf{1}_{(h_k, h_{k+1}]} \frac{Q_k}{R_k}. \quad (1)$$

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To simplify notation in Eq. (1), for $k = K-1$ we write $(h_{K-1}, h_K]$ and not (h_{K-1}, h_K). Our main result is the following.

Theorem 2.3. *The set $\mathcal{V}$ coincides with the set of all continuous and piecewise rational functions from $\mathbf{R}$ to $\mathbf{R}$.*

3. Proof of Theorem 2.3

We start by proving the necessity of the condition. Let G be a polynomial game. Since the function that assigns to every two-player zero-sum game its value is continuous, it follows that the function $z \mapsto \mathrm{val}(\Gamma(z))$ is continuous.

For every square matrix H, denote by $\det(H)$ its determinant, by $\mathrm{co}(H)$ the cofactor matrix whose (i,j) entry is the determinant of the submatrix of H obtained by deleting the i'th row and the j'th column, and by $S(H)$ the sum of all elements of H. By [6], for every $z \in \mathbf{R}$ there is a square submatrix $H(z)$ of G such that $\mathrm{val}(\Gamma(z)) = \frac{\det(H(z))}{S(\mathrm{co}(H(z)))}$. Since the number of square submatrices of G is finite, since for every square submatrix H of G the function $z \mapsto \frac{\det(H(z))}{S(\mathrm{co}(H(z)))}$ is a rational function, and since two distinct rational functions intersect in finitely many points, it follows that the function $z \mapsto \mathrm{val}(\Gamma(z))$ is piecewise rational.

We turn to prove the sufficiency of the condition. To this end we prove the following simple properties of the set $\mathcal{V}$.

Proposition 3.1. *Let $u, w \in \mathcal{V}$. Then*

(A.1) $-u \in \mathcal{V}$.
(A.2) $u + w \in \mathcal{V}$.
(A.3) $\max\{u, w\} \in \mathcal{V}$.
(A.4) *If there is $z_0 \in \mathbf{R}$ such that $u(z_0) = w(z_0)$ then the following function $v : \mathbf{R} \to \mathbf{R}$ is in $\mathcal{V}$:*

$$v(z) := \mathbf{1}_{z \le z_0} u(z) + \mathbf{1}_{z > z_0} w(z), \quad \forall z \in \mathbf{R}. \tag{2}$$

(A.5) $P \cdot u \in \mathcal{V}$ *for every polynomial P.*
(A.6) *If there is $\varepsilon > 0$ such that $u(z) \ge \varepsilon$ for every $z \in \mathbf{R}$, then $\frac{1}{u} \in \mathcal{V}$.*

Proof. Since $u, w \in \mathcal{V}$, there exist polynomial games $\Gamma_u = (A_u, B_u, G_u)$ and $\Gamma_w = (A_w, B_w, G_w)$ such that u is the value function of Γ_u and w is the value function of Γ_w.

By changing the role of the two players, the value of the game is multiplied by -1. That is, the value of the game $(B_u, A_u, (-G_u)^T)$ is minus the value of the game (A_u, B_u, G_u). (A.1) follows.

To see that (A.2) holds, suppose that the two players simultaneously play the games Γ_u and Γ_w, and the payoff is the sum of the payoffs in the two games. Formally, we consider the game $\Gamma = (A, B, G)$ where $A = A_u \times A_w$, $B = B_u \times B_w$, and $G_{(a_u,a_w),(b_u,b_w)} = G_{u,a_u,b_u} + G_{w,a_w,b_w}$. The reader can verify that for every $z \in \mathbf{R}$, $\mathrm{val}(\Gamma(z)) = \mathrm{val}(\Gamma_u(z)) + \mathrm{val}(\Gamma_w(z))$.

To see that (A.3) holds, suppose again that the two players simultaneously play the games Γ_u and Γ_w, and, when choosing his strategies in the two games, Player 1 also chooses whether the payoff will be the payoff in Γ_u or in Γ_w. Formally, we consider the game $\Gamma = (A, B, G)$ where $A = A_u \times A_w \times \{U, W\}$, $B = B_u \times B_w$, $G_{(a_u,a_w,U),(b_u,b_w)} = G_{u,a_u,b_u}$, and $G_{(a_u,a_w,W),(b_u,b_w)} = G_{w,a_w,b_w}$. The reader can verify that for every $z \in \mathbf{R}$, $\mathrm{val}(\Gamma(z)) = \max\{\mathrm{val}(\Gamma_u(z)), \mathrm{val}(\Gamma_w(z))\}$.

We next prove that (A.4) holds. For every two rational functions $\widehat{u}, \widehat{w} : \mathbf{R} \to \mathbf{R}$ such that $\widehat{u}(0) = \widehat{w}(0) = 0$ there are $c > 0$ and $N \in \mathbf{N}$ sufficiently large such that $\widehat{u}(z) \ge cz + z^{2N+1}$ for every $z \le 0$, and $\widehat{w}(z) \le cz + z^{2N+1}$ for every $z \ge 0$. Consequently, since u and w are continuous piecewise rational functions, there are two polynomials $P_1, P_2 : \mathbf{R} \to \mathbf{R}$ such that

- $u(z) \ge P_1(z)$ for every $z \le z_0$, and $u(z), w(z) \le P_1(z)$ for every $z \ge z_0$.
- $u(z), w(z) \le P_2(z)$ for every $z \le z_0$, and $w(z) \ge P_2(z)$ for every $z \ge z_0$.

The reader can verify that the function v that is defined in Eq. (2) coincides with $\min\{\max\{u, P_1\}, \max\{w, P_2\}\}$, and (A.4) follows from (A.1) and (A.3).

To prove (A.5), fix a polynomial P. Denote by $\Gamma_u^P(A_u, B_u, G_u^P)$ the polynomial game that is defined by $G_{u,a,b}^P := P \cdot G_{u,a,b}$ for every $(a, b) \in A_u \times B_u$. For every $z \in \mathbf{R}$ for which $P(z) \ge 0$ we have $\mathrm{val}(\Gamma_u^P(z)) = P(z)u(z)$. By a repeated use of (A.4), the following function u_1 is in $\mathcal{V}$:

$$u_1(z) := \begin{cases} P(z) \cdot u(z) & \text{if } P(z) \ge 0, \\ 0 & \text{if } P(z) < 0. \end{cases}$$

By (A.1) we have $-u \in \mathcal{V}$, hence as above the function u_2 that is defined by

$$u_2(z) := \begin{cases} (-P(z)) \cdot (-u(z)) & \text{if } P(z) \le 0, \\ 0 & \text{if } P(z) > 0, \end{cases}$$

is in $\mathcal{V}$. Since $u = u_1 + u_2$, (A.5) follows from (A.2).

We finally turn to prove that (A.6) holds. Let $\Gamma = (A, B, G)$ be the game where $A = \{1\} \cup A_u$, $B = \{1\} \cup B_u$, and

$$G = \begin{Bmatrix} \varepsilon & \mathbf{0}^{1 \times |B_u|} \\ \mathbf{0}^{|A_u| \times 1} & G_u - \boldsymbol{\varepsilon}^{|A_u| \times |B_u|} \end{Bmatrix},$$

where $\mathbf{0}^{1 \times |B_u|}$ and $\mathbf{0}^{|A_u| \times 1}$ are matrices of sizes $1 \times |B_u|$ and $|A_u| \times 1$ all of whose elements are 0, and $\boldsymbol{\varepsilon}^{|A_u| \times |B_u|}$ is the matrix of size $|A_u| \times |B_u|$ all of whose elements are ε. The value of the strategic-form game $(A_u, B_u, G_u(z) - \boldsymbol{\varepsilon}^{|A| \times |B|})$ is $u(z) - \varepsilon$, which is positive. Since $\varepsilon > 0$, it follows that the value of the strategic-form game $(A, B, G(z))$ is the same as the value of the 2×2 strategic-form game $\begin{Bmatrix} \varepsilon & 0 \\ 0 & u(z) - \varepsilon \end{Bmatrix}$, which is $\frac{\varepsilon(u(z)-\varepsilon)}{u(z)} = \varepsilon - \frac{\varepsilon^2}{u}$. (A.6) follows. ■

We now have the tools to prove that every continuous piecewise rational function is in $\mathcal{V}$. Note that since (A.4) does not hold for one-player polynomial games, there is a continuous piecewise rational function that is not the value function of any one-player polynomial game.

Proposition 3.2. *Let $u : \mathbf{R} \to \mathbf{R}$ be a continuous function that is piecewise rational. Then $u \in \mathcal{V}$.*

Proof. Since the function u is piecewise rational, there are $K \in \mathbf{N}$, $-\infty = h_1 < h_2 < \cdots < h_K = \infty$, and rational functions $\left(\frac{Q_k}{R_k}\right)_{k=0}^{K-1}$ such that $u = \frac{Q_k}{R_k}$ on the interval (h_k, h_{k+1}) and Q_k and R_k do not have common roots, for every k. Assume w.l.o.g. that for every k the polynomial R_k is positive on (h_k, h_{k+1}), and denote by $\varepsilon_k := \inf\{R_k(z) : z \in (h_k, h_{k+1})\}$ the minimum of R_k in this interval. Since $u(z)$ is finite for every $z \in \mathbf{R}$, and since R_k and Q_k do not have common roots, it follows that R_k has no root in the closure of (h_k, h_{k+1}), for every k.

We argue that $\varepsilon_k > 0$ for every k. For $k \ne 1, K-1$ this holds since the interval $[h_k, h_{k+1}]$ is compact. For $k = 1$ we have $\varepsilon_k > 0$ since $\lim_{z \to -\infty} R_1(z) > 0$ and since $R_1(z) > 0$ on $(-\infty, h_1]$. For analogous reasons, $\varepsilon_{K-1} > 0$.

By (A.3) we have $\max\{R_k, \varepsilon_k\} \in \mathcal{V}$ for every k. By (A.5) and (A.6) and since $\varepsilon_k > 0$ we have $\frac{Q_k}{\max\{R_k, \varepsilon_k\}} \in \mathcal{V}$ for every k. By iterative use of (A.4), we have $\sum_{k=1}^{K-1} \mathbf{1}_{z \in (h_k, h_{k+1}]} \frac{Q_k}{\max\{R_k, \varepsilon_k\}} \in \mathcal{V}$. Since $R_k \ge \varepsilon_k$ on (h_k, h_{k+1}), for every k, this function is equal to u, and the result follows. ■

4. Discussion and open problems

Let $\mathcal{F}$ be the family of all polynomial games where the entries of the payoff matrix are polynomials with rational coefficients. A corollary of our result is that the set $\mathcal{V}_{\mathcal{F}}$ of value functions of polynomial games in $\mathcal{F}$ coincides with the set of continuous piecewise rational functions $u \in \mathcal{V}$ where, in the presentation (1), the polynomials $(Q_k, R_k)_{k=0}^{K-1}$ have integer coefficients with algebraic $(h_i)_{i=1}^{K-1}$.

A *piecewise rational game* (resp. an *affine game*) is a two-player game where each entry in the payoff matrix is a continuous piecewise rational (resp. affine) function of the parameter Z. The arguments that we provided for the necessary condition imply that the set of value functions of piecewise rational games coincides with $\mathcal{V}$. This implies that even though the payoff entries of piecewise rational games can be more complicated than those of polynomial games, for any piecewise rational game one can construct a polynomial game with the same value function. In contrast, the set of value functions of affine games is a strict subset of $\mathcal{V}$. Indeed, in an affine game the maximal payoff is linear in Z, hence the value function is bounded by an affine function of Z. In particular, the function Z^2 cannot be the value function of an affine game.

When the domain of the parameter Z is a compact interval I rather than $\mathbf{R}$, the set $\mathcal{V}_I$ of value functions of affine games coincides with the restriction of $\mathcal{V}$ to I; that is, the set $\mathcal{V}_I$ is the set of all continuous and piecewise rational functions from I to $\mathbf{R}$. The only parts in our proof that do not carry over to affine games are the proofs of Conditions (A.4) and (A.5). In the proof of (A.4) we use the evident fact that $\mathcal{V}$ contains all polynomials. For affine games this property is no longer evident. Yet, one can prove by induction on k that $Z^k \in \mathcal{V}_I$, for every $k \in \mathbf{N}$. In the proof of (A.5) we use the fact that the product of polynomials remains a polynomial. However, the product of affine functions is not necessarily an affine function. Instead of proving (A.5), one can prove by induction on the degree of Q that for every positive real number $\varepsilon > 0$ and every two polynomials Q and R, the function $\frac{Q}{\max\{R,\varepsilon\}} \in \mathcal{V}_I$, which is the property that is needed in the proof of Proposition 3.2.

One example of an affine game that is defined over a compact interval is a Bayesian game with two states of nature s_0 and s_1, where the parameter Z is the prior probability that the state is s_0. Our result implies that the set of all value functions of Bayesian games with two states of nature is the set of all continuous piecewise rational functions defined on $[0, 1]$.

When $u = \sum_{k=0}^{K-1} \mathbf{1}_{(h_k,h_{k+1}]} \frac{Q_k}{R_k}$ is a continuous piecewise rational function, our construction shows that there is a polynomial game $\Gamma_u = (A_u, B_u, G_u)$ whose value function is u and such that the number of actions of both players is of the order of 2^K. In addition, the degree of the polynomials in the payoff matrix G_u is at most $\max_k\{\deg(Q_k) + \deg(R_k) + 1\}$. We do not know whether one can improve upon these bounds.

A natural question that arises from our study is what happens when the payoffs depend on two (or more) parameters, say Z and W. In that case, the value function is a function that assigns a real number to every $(Z, W) \in \mathbf{R}^2$. Some parts of our arguments extend to this case. The difficult step is to prove that (A.4) holds. This condition turns out to be related to the Pierce–Birkhoff Conjecture [1], which asks whether every continuous piecewise polynomial function in $\mathbf{R}^d$ is the maximum of finitely many minima of finitely many polynomials.

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