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THE VALUE OF ZERO-SUM STOPPING GAMES IN CONTINUOUS TIME*

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Abstract. We study two-player zero-sum stopping games in continuous time and infinite horizon. We prove that the value in randomized stopping times exists as soon as the payoff processes are right-continuous. In particular, as opposed to existing literature, we do *not* assume any conditions on the relations between the payoff processes.

Key words. Dynkin games, stopping games, optimal stopping, stochastic analysis, continuous time, stochastic duels

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1. Introduction.

In many competitive interactions the main strategic issue is timing. To model such situations, Dynkin (1969) introduced stopping games, as a variation of optimal stopping problems. In Dynkin's setup, two players observe the realization of a payoff process in discrete time. Once one of the players decides to stop, player 2 pays player 1 the amount indicated by the payoff process. However, at every given stage only one of the players is allowed to stop; the identity of that player is governed by another process. The strategic choice of each player is the choice of his stopping time. Dynkin (1969) proved that those games admit a value.

Dynkin's seminal paper was extended in various directions. Neveu (1975) allowed the players to stop *simultaneously* and provided a sufficient condition for the existence of the value. Several authors, including Bismut (1977), Alario-Nazaret, Lepeltier, and Marchal (1982), Lepeltier and Maingueneau (1984), and Stettner (1984) studied the problem in *continuous time*.

Yasuda (1985) studied stopping games in discrete time (with either finite horizon or discounted payoff), and allowed the players to choose *randomized* stopping times. Yasuda (1985) proved that the value exists under merely an integrability condition. Rosenberg, Solan, and Vieille (2001) studied the infinite horizon game in discrete time and proved an analogous result. Touzi and Vieille (2002) provided a sufficient condition that ensures the existence of the value in randomized stopping times in continuous time. As their proof utilizes a fixed point argument, it is not constructive.

In the present paper we prove that under merely integrability and continuity conditions, every stopping game in continuous time admits a value in randomized stopping times. In addition, we construct ε-optimal randomized stopping times which are as close as one wishes to (nonrandomized) stopping times; roughly speaking, there

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is a stopping time μ such that for every δ sufficiently small there is an ε-optimal randomized stopping time that stops with probability 1 between times μ and $\mu + \delta$.

Several models that have been extensively studied in different disciplines and that fall into the category of stopping games are wars of attrition (see, e.g., Maynard-Smith (1974), Ghemawat and Nalebuff (1985), and Hendricks, Weiss, and Wilson (1988)), preemption games (see, e.g., Fudenberg and Tirole (1991, section 4.5.3)), duels, and pricing of options. We will illustrate the applicability of our results by discussing the last two models.

We first present the model of duels. In the simplest version, duels are two-player zero-sum games in which each of two gunners is endowed with a single bullet. The two gunners are located at some distance from each other and move closer to one another as time goes on. Since the accuracy of their shots improves as they get closer, it is not clear what the optimal moment is to shoot the opponent. If the accuracy is a stochastic process that depends on, e.g., wind velocity, the gunners face a stopping game.

Although for various classes of duels the existence of the value has been established, and optimal strategies have been computed (see, e.g., Blackwell (1949), Bellman and Girshick (1949), Shapley (1951), Karlin (1959), and the recent survey by Radzik and Raghavan (1994)), the general case is still open.

As we argue below, our results can be applied to any duel, regardless of the number of bullets each player initially has.

We now discuss the relevant literature in pricing of options. In most cases, a holder of an option has the right to exercise the option either on prespecified dates or whenever he chooses, so that the optimization problem reduces to an optimal stopping problem. Callable warrants (see, e.g., Merton (1973)) and convertible bonds (see, e.g., Brennan and Schwartz (1977)) allow for a certain action by the issuer as well. Recently Kifer (2000) introduced game contingent claims, which are general American options in which the issuer can terminate the contract early at some cost. Kifer showed that pricing these options boils down to determining the value of a certain stopping game, and he provided a general characterization for the value. Game contingent claims have been studied also by, e.g., Kallsen and Kühn (2004) and Kühn and Kyprianou (2003). Kyprianou (2004) used Kifer's characterization to explicitly calculate the value of game contingent claims in some cases. McConnell and Schwartz (1986) studied a specific example of callable option notes, which were actually issued in the 1980s.

In the formulation of game contingent claims in Kifer (2000), the right of the holder to exercise the option supersedes the right of the issuer to terminate the contract early, so that if those two events occur simultaneously, the holder gets to exercise the option. However, if the payment when those two events occur simultaneously is different from the payment if the holder were to exercise alone, or the issuer were to terminate the contract alone, Kifer's analysis would no longer be valid. Our result establishes the existence of the value in this case, and may be used, as was done by Kyprianou (2004), to find optimal strategies in given examples.

The paper is arranged as follows. The model and the main result appear in section 2, and the proof of the main result appears in section 3. Further topics, namely, introducing final payoffs and the right-continuity of the value process, are discussed in sections 3.4–3.5. We end by using the right-continuity of the value process to derive a more general existence result for noisy stochastic duels in section 3.6.

2. Model, literature, and main result.

A *two-player zero-sum stopping game in continuous time* Γ is given by the following:

- A probability space $(\Omega, \mathcal{A}, P)$: $(\Omega, \mathcal{A})$ is a measurable space, and P is a σ-additive probability measure on $(\Omega, \mathcal{A})$.
- A filtration in continuous time $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying "the usual conditions." That is, $\mathcal{F}$ is right-continuous, and $\mathcal{F}_0$ contains all P-null sets: for every $B \in \mathcal{A}$ with $P(B) = 0$ and every $A \subset B$, one has $A \in \mathcal{F}_0$. All stopping times in what follows are of the filtration $\mathcal{F}$.
  Denote $\mathcal{F}_\infty := \vee_{t\geq 0}\mathcal{F}_t$. Assume without loss of generality that $\mathcal{F}_\infty = \mathcal{A}$. Hence $(\Omega, \mathcal{A}, P)$ is a complete probability space.
- Three uniformly bounded $\mathcal{F}$-adapted processes $(a_t, b_t, c_t)_{t\geq 0}$.[1]

DEFINITION 1. *A* randomized stopping time *is a progressively measurable function $\phi : [0,1] \times \Omega \to [0, +\infty]$ such that for every $r \in [0,1]$, $\mu_r(\omega) := \phi(r, \omega)$ is an optional stopping time.*

For strategically equivalent definitions of randomized stopping times, see Touzi and Vieille (2002). Throughout the paper, the symbols μ and ν stand for stopping times, while ϕ and ψ stand for randomized stopping times.

For every pair (μ, ν) of stopping times we denote

$$\gamma(\mu, \nu) = \mathbf{E}_P\left[a_\mu \mathbf{1}_{\{\mu<\nu\}} + b_\nu \mathbf{1}_{\{\mu>\nu\}} + c_\mu \mathbf{1}_{\{\mu=\nu<+\infty\}}\right].$$

The *expected payoff* that corresponds to a pair of randomized stopping times (ϕ, ψ) is

$$\begin{aligned}\gamma(\phi, \psi) &= \int_{[0,1]^2} \gamma(\mu_r, \nu_s)\, dr\, ds \\ &= \mathbf{E}_{\lambda\otimes\lambda\otimes P}\left[a_{\mu_r}\mathbf{1}_{\{\mu_r<\nu_s\}} + b_{\nu_s}\mathbf{1}_{\{\mu_r>\nu_s\}} + c_{\mu_r}\mathbf{1}_{\{\mu_r=\nu_s<+\infty\}}\right].\end{aligned} \tag{1}$$

Though the payoff function given by (1) is bilinear, without strong assumptions on the data of the game, the payoff function is not continuous for the same topology which makes the space of randomized stopping times compact.

DEFINITION 2. *If $\sup_\phi \inf_\psi \gamma(\phi, \psi) = \inf_\psi \sup_\phi \gamma(\phi, \psi)$, then the common value is the* value in randomized stopping times *and is denoted by V. Every randomized stopping time ϕ such that $\inf_\psi \gamma(\phi, \psi)$ is within ε of V is* ε-optimal *for player* 1*; ε-optimal randomized stopping times for player* 2 *are defined analogously.*

Observe that for every ϕ one has $\inf_\psi \gamma(\phi, \psi) = \inf_\nu \gamma(\phi, \nu)$, where ν ranges over all stopping times. Indeed, for every ϕ and ψ one has, by Fubini's theorem,

$$\gamma(\phi, \psi) = \mathbf{E}_{\lambda\otimes\lambda\otimes P}[\gamma(\mu_r, \nu_s)] \geq \inf_s \mathbf{E}_{\lambda\otimes P}[\gamma(\mu_r, \nu_s)] \geq \inf_\nu \gamma(\phi, \nu) \geq \inf_{\psi'} \gamma(\phi, \psi').$$

This implies that $\sup_\phi \inf_\psi \gamma(\phi, \psi) = \sup_\phi \inf_\nu \gamma(\phi, \nu)$. Similarly, $\inf_\psi \sup_\phi \gamma(\phi, \psi) = \inf_\psi \sup_\mu \gamma(\mu, \psi)$, where μ ranges over all pure stopping times. Recall that one always has $\sup_\phi \inf_\psi \gamma(\phi, \psi) \leq \inf_\psi \sup_\phi \gamma(\phi, \psi)$.

Touzi and Vieille (2002) provided a restrictive condition that ensures the existence of the value. The main result we present is the following.

THEOREM 3. *If the processes $(a_t)_{t\geq 0}$ and $(b_t)_{t\geq 0}$ are right-continuous, and if $(c_t)_{t\geq 0}$ is progressively measurable, then the value in randomized stopping times exists.*

3. Proof of the main result and extensions.

From now on we fix a stopping game Γ that satisfies the assumptions of Theorem 3.

[1] Our results hold for a larger class of payoff processes, namely, the class $\mathcal{D}$ that was defined by Dellacherie and Meyer (1975, section II-18). This class contains in particular integrable processes.

3.1. Preliminaries. The following lemma will be used in what follows.

LEMMA 4. *For every stopping time τ and every $\varepsilon > 0$ there is $\delta > 0$ such that $P(\{|a_t - a_\tau| < \varepsilon \quad \forall t \in [\tau, \tau + \delta]\}) > 1 - \varepsilon$.*

A similar statement holds when one replaces the process $(a_t)_{t\geq 0}$ by the process $(b_t)_{t\geq 0}$.

Proof. For every $n \in \mathbf{N}$, set $A_n = \{\sup\{|a_t - a_\tau|, \tau \leq t \leq \tau + 1/n\} < \varepsilon\}$. Since $(a_t)_{t\geq 0}$ is right-continuous, $P(\cup_{n\in\mathbf{N}} A_n) = 1$, and the result follows. □

One then obtains the following result.

COROLLARY 5. *Let a stopping time τ and $\varepsilon > 0$ be given. There exists $\delta > 0$ sufficiently small such that for every $\mathcal{F}_\tau$-measurable set $A \subseteq \{\tau < +\infty\}$ and every stopping time μ that satisfies $\tau \leq \mu \leq \tau + \delta$,*

$$|\mathbf{E}_P[a_\mu \mathbf{1}_A] - \mathbf{E}_P[a_\tau \mathbf{1}_A]| \leq \varepsilon.$$

3.2. The case $a_t \leq b_t$ for every $t \geq 0$. In this section we prove the following result: when $a_t \leq b_t$ for every $t \geq 0$, the value in randomized stopping times exists and is independent of $(c_t)_{t\geq 0}$.

The idea is as follows. Assume player 1 decides to stop at time t_*. If $c_{t_*} \geq a_{t_*}$, and player 1 stops with probability 1 at time t_*, player 2 has no incentive to stop at t_* as well. However, if $c_{t_*} < a_{t_*}$, player 1 needs to mask the exact time in which he stops, so that player 2 cannot stop at the same time. Since payoffs are right-continuous, player 1 can stop randomly in a small interval after time t_*. This way he makes sure that player 2 does not know the exact moment he will stop, and since $a_t \leq b_t$ for every t, player 2 has no incentive to stop in this time interval. In both cases, whatever the process $(c_t)_{t\geq 0}$ is, if the game has not stopped before time t_* player 1 can guarantee a payoff close to a_{t_*}.

PROPOSITION 6. *If $a_t \leq b_t$ for every $t \geq 0$, then the value in randomized stopping times exists. Moreover, the value is independent of the process $(c_t)_{t\geq 0}$. If $a_t \leq c_t \leq b_t$ for every $t \geq 0$, then there are ε-optimal (nonrandomized) stopping times for both players that are independent of $(c_t)_{t\geq 0}$.*

Proof. Consider an auxiliary stopping game $\Gamma^* = (\Omega, \mathcal{A}, P; \mathcal{F}, (a^*_t, b^*_t, c^*_t)_{t\geq 0})$, where $a^*_t = a_t$ and $b^*_t = c^*_t = b_t$ for every $t \geq 0$.

Lepeltier and Maingueneau (1984) and Stettner (1984) proved that the game Γ^* admits a value, and that there are ε-optimal (nonrandomized) stopping times for both players. We denote the value of Γ^* by v^* and prove that it is the value in randomized stopping times of the original game. Since Γ^* does not depend on the process $(c_t)_{t\geq 0}$, the second assertion of the proposition follows.

Fix $\varepsilon > 0$. Let μ be an ε-optimal (nonrandomized) stopping time for player 1 in Γ^*. In particular, $\inf_\nu \gamma_{\Gamma^*}(\mu, \nu) \geq v^* - \varepsilon$.

We now construct a randomized stopping time ϕ that satisfies $\inf_\nu \gamma_\Gamma(\phi, \nu) \geq v^* - 3\varepsilon$. By Lemma 4 there is $\delta > 0$ such that $P(\{|a_t - a_\mu| < \varepsilon \quad \forall t \in [\mu, \mu + \delta]\}) > 1 - \varepsilon$. Define a randomized stopping time ϕ by

$$\phi(r, \cdot) = \mu + r\delta \qquad \forall r \in [0, 1].$$

That is, ϕ stops at a random time in the interval $[\mu, \mu + \delta]$. We denote such a randomized stopping time by $\phi = \mu + r\delta$.

Let ν be any stopping time. Since μ is ε-optimal in Γ^*, by the definition of Γ^*, by Fubini's theorem, and since $\lambda \otimes P(\mu + r\delta = \nu) = 0$,

$$
\begin{aligned}
v^* - \varepsilon &\leq \gamma_{\Gamma^*}(\mu, \nu) \\
&= \mathbf{E}_P[a_\mu \mathbf{1}_{\{\mu < \nu\}} + b_\nu \mathbf{1}_{\{\mu \geq \nu\}}] \\
&= \mathbf{E}_{\lambda \otimes P}[a_\mu \mathbf{1}_{\{\mu + r\delta < \nu\}} + a_\mu \mathbf{1}_{\{\mu < \nu < \mu + r\delta\}} + b_\nu \mathbf{1}_{\{\mu \geq \nu\}}].
\end{aligned} \tag{2}
$$

Since $\lambda \otimes P(\mu + r\delta = \nu) = 0$ and $(c_t)_{t \geq 0}$ is progressively measurable,

$$
\begin{aligned}
\gamma_\Gamma(\phi, \nu) &= \mathbf{E}_{\lambda \otimes P}\left[a_{\mu + r\delta} \mathbf{1}_{\{\mu + r\delta < \nu\}} + b_\nu \mathbf{1}_{\{\mu + r\delta > \nu\}} + c_\nu \mathbf{1}_{\{\mu + r\delta = \nu < +\infty\}}\right] \\
&= \mathbf{E}_{\lambda \otimes P}\left[a_{\mu + r\delta} \mathbf{1}_{\{\mu + r\delta < \nu\}} + b_\nu \mathbf{1}_{\{\mu + r\delta > \nu\}}\right] \\
&= \mathbf{E}_{\lambda \otimes P}\left[a_{\mu + r\delta} \mathbf{1}_{\{\mu + r\delta < \nu\}} + b_\nu \mathbf{1}_{\{\mu < \nu < \mu + r\delta\}} + b_\nu \mathbf{1}_{\{\mu \geq \nu\}}\right].
\end{aligned} \tag{3}
$$

By Corollary 5, and since $a_t \leq b_t$ for every $t \geq 0$,

$$
\mathbf{E}_{\lambda \otimes P}[a_\mu \mathbf{1}_{\{\mu < \nu < \mu + r\delta\}}] \leq \mathbf{E}_{\lambda \otimes P}[a_\nu \mathbf{1}_{\{\mu < \nu < \mu + r\delta\}}] + \varepsilon \leq \mathbf{E}_{\lambda \otimes P}[b_\nu \mathbf{1}_{\{\mu < \nu < \mu + r\delta\}}] + \varepsilon. \tag{4}
$$

Corollary 5 implies in addition that for every $r \in [0, 1]$

$$
\mathbf{E}_{\lambda \otimes P}[a_\mu \mathbf{1}_{\{\mu + r\delta < \nu\}}] \leq \mathbf{E}_{\lambda \otimes P}[a_{\mu + r\delta} \mathbf{1}_{\{\mu + r\delta < \nu\}}] + \varepsilon. \tag{5}
$$

By (2)–(5),

$$
v^* - \varepsilon \leq \gamma_{\Gamma^*}(\mu, \nu) \leq \gamma_\Gamma(\phi, \nu) + 2\varepsilon.
$$

Since ν is arbitrary, $\inf_\nu \gamma_\Gamma(\phi, \nu) \geq v^* - 3\varepsilon$.

Consider a second auxiliary stopping game $\Gamma^{**} = (\Omega, \mathcal{A}, P; \mathcal{F}, (a_t^{**}, b_t^{**}, c_t^{**})_{t \geq 0})$, where $a_t^{**} = c_t^{**} = a_t$ and $b_t^{**} = b_t$ for every $t \geq 0$.

A symmetric argument to the one provided above proves that the game Γ^{**} has a value v^{**} and that player 2 has a randomized stopping time ψ that satisfies $\sup_\mu \gamma_\Gamma(\mu, \psi) \leq v^{**} + 3\varepsilon$.

Since $c_t^{**} = a_t \leq b_t = c_t^*$ for every $t \geq 0$, one has $v^{**} \leq v^*$. Since $\sup_\mu \gamma_\Gamma(\mu, \psi) \geq \gamma_\Gamma(\phi, \psi) \geq \inf_\nu \gamma_\Gamma(\phi, \nu)$,

$$
v^* \geq v^{**} \geq \sup_\mu \gamma_\Gamma(\mu, \psi) - 3\varepsilon \geq \inf_\nu \gamma_\Gamma(\phi, \nu) - 3\varepsilon \geq v^* - 6\varepsilon.
$$

Since ε is arbitrary, $v^* = v^{**}$, so that v^* is the value in randomized stopping times of Γ, and ϕ and ψ are 3ε-optimal randomized stopping times for the two players. The first assertion of the proposition is established.

We now turn to the third assertion of the proposition. If $a_t \leq c_t \leq b_t$ for every $t \geq 0$, then $\gamma_{\Gamma^{**}}(\mu, \nu) \leq \gamma_\Gamma(\mu, \nu) \leq \gamma_{\Gamma^*}(\mu, \nu)$ for every pair of stopping times (μ, ν). Hence

$$
\begin{aligned}
v^{**} &= \sup_\mu \inf_\nu \gamma_{\Gamma^{**}}(\mu, \nu) \leq \sup_\mu \inf_\nu \gamma_\Gamma(\mu, \nu) \\
&\leq \inf_\nu \sup_\mu \gamma_\Gamma(\mu, \nu) \leq \inf_\nu \sup_\mu \gamma_{\Gamma^*}(\mu, \nu) = v^* = v^{**}.
\end{aligned}
$$

Thus $\sup_\mu \inf_\nu \gamma_\Gamma(\mu, \nu) = \inf_\nu \sup_\mu \gamma_\Gamma(\mu, \nu)$: the value exists and there are ε-optimal stopping times for both players. Moreover, any ε-optimal stopping time for player 1 (resp., player 2) in Γ^* (resp., Γ^{**}) is also ε-optimal in Γ. In particular, if $a_t \leq c_t \leq b_t$ for every $t \geq 0$, both players have ε-optimal stopping times that are independent of $(c_t)_{t \geq 0}$. $\square$

3.3. Proof of Theorem 3. Define a stopping time τ by

$$\tau = \inf\{t \geq 0, a_t \geq b_t\},$$

where the infimum of an empty set is $+\infty$. Since $(a_t - b_t)_{t\geq 0}$ is progressively measurable with respect to $(\mathcal{F}_t)_{t\geq 0}$, τ is a stopping time (see, e.g., Dellacherie and Meyer (1975, section IV-50)).

We show below that it is optimal for both players to stop at or around time τ (provided the game does not stop before time τ). Hence the problem reduces to the game between times 0 and τ. Since for $t \in [0, \tau[$, $a_t \leq b_t$, Proposition 6 can be applied.

The following notation will be useful in what follows. For a pair of stopping times (μ, ν) and a set $A \in \mathcal{A}$ we define

$$\gamma_\Gamma(\mu, \nu; A) = \mathbf{E}_P[\mathbf{1}_A(a_\mu \mathbf{1}_{\{\mu<\nu\}} + b_\mu \mathbf{1}_{\{\mu>\nu\}} + c_\mu \mathbf{1}_{\{\mu=\nu<+\infty\}})].$$

This is the expected payoff restricted to A. For a pair of randomized stopping times (ϕ, ψ) we define

$$\gamma_\Gamma(\phi, \psi; A) = \int_{[0,1]^2} \gamma_\Gamma(\mu_r, \nu_s; A) dr\ ds,$$

where μ_r and ν_s are the sections of ϕ and ψ, respectively.

Set

$$\begin{aligned} A_0 &= \{\tau = +\infty\}, \\ A_1 &= \{\tau < +\infty\} \cap \{c_\tau \geq a_\tau \geq b_\tau\}, \\ A_2 &= \{\tau < +\infty\} \cap \{a_\tau > c_\tau \geq b_\tau\}, \text{ and} \\ A_3 &= \{\tau < +\infty\} \cap \{a_\tau \geq b_\tau > c_\tau\}. \end{aligned}$$

Observe that (A_0, A_1, A_2, A_3) is an $\mathcal{F}_\tau$-measurable partition of Ω.

Define an $\mathcal{F}_\tau$-measurable function w by

$$w = a_\tau \mathbf{1}_{A_1} + c_\tau \mathbf{1}_{A_2} + b_\tau \mathbf{1}_{A_3}.$$

Define a stopping game $\Gamma^* = (\Omega, \mathcal{A}, P, (\mathcal{F}_t)_{t\geq 0}, (a_t^*, b_t^*, c_t^*)_{t\geq 0})$ by

$$a_t^* = \begin{cases} a_t & t < \tau \\ w & t \geq \tau \end{cases}, \qquad b_t^* = \begin{cases} b_t & t < \tau \\ w & t \geq \tau \end{cases}, \qquad c_t^* = \begin{cases} c_t & t < \tau \\ w & t \geq \tau \end{cases}.$$

That is, the payoff is set to w at and after time τ.

The game Γ^* satisfies the assumptions of Proposition 6 and hence, has a value V in randomized stopping times.

We now prove that V is the value of the game Γ as well. Fix $\varepsilon > 0$. We show only that player 1 has a randomized stopping time ϕ such that $\inf_\nu \gamma_\Gamma(\phi, \nu) \geq V - 7\varepsilon$. An analogous argument shows that player 2 has a randomized stopping time ψ such that $\sup_\mu \gamma_\Gamma(\mu, \psi) \leq V + 7\varepsilon$. Since ε is arbitrary, V is indeed the value in randomized stopping times of Γ.

Assume δ is sufficiently small so that the following conditions hold (by the proofs of Lemma 4 and Proposition 6 such δ exists):

(C1) There exists a stopping time μ such that the randomized stopping time $\phi^* = \mu + r\delta$ is ε-optimal for player 1 in Γ^*.

(C2) $P(\{\mu+\delta<\tau\}) \geq P(\{\mu<\tau\}) - \varepsilon/M$, where $M \in]0,+\infty[$ is a uniform bound of the payoff processes.

(C3) $P(\{|a_t - a_\tau| < \varepsilon, |b_t - b_\tau| < \varepsilon \quad \forall t \in [\tau, \tau+\delta]\}) > 1-\varepsilon$.

We now claim that we can assume without loss of generality that $\mu \leq \tau$. Indeed, assume that $P(\{\mu > \tau\}) > 0$. The set $\{\mu > \tau\}$ is $\mathcal{F}_\tau$-measurable. Define a stopping time $\mu' = \min\{\mu, \tau\}$. We will prove that the randomized stopping time $\phi' = \mu' + r\delta$ is also ε-optimal in Γ^*, which establishes the claim. Given a stopping time ν define a stopping time ν' by $\nu' = \min\{\nu, \tau\}$. By (C1),

$$\begin{aligned} V - \varepsilon &\leq \gamma_{\Gamma^*}(\mu + r\delta, \nu') \\ &= \gamma_{\Gamma^*}(\mu + r\delta, \nu'; \{\mu > \tau\}) + \gamma_{\Gamma^*}(\mu + r\delta, \nu'; \{\mu \leq \tau < \mu + \delta\}) \\ &\quad + \gamma_{\Gamma^*}(\mu + r\delta, \nu'; \{\mu + \delta \geq \tau\}). \end{aligned}$$

On the right-hand side the first term is equal to $\gamma_{\Gamma^*}(\mu' + r\delta, \nu; \{\mu > t\})$, by (C2) the second term is bounded by ε, and the third term is equal to $\gamma_{\Gamma^*}(\mu' + r\delta, \nu; \{\mu + \delta \geq \tau\})$. Therefore, by (C2),

$$\begin{aligned} V - \varepsilon &\leq \gamma_{\Gamma^*}(\mu' + r\delta, \nu; \{\mu > t\}) + \varepsilon + \gamma_{\Gamma^*}(\mu' + r\delta, \nu; \{\mu + \delta \geq \tau\}) \\ &\leq \gamma_{\Gamma^*}(\mu' + r\delta; \nu) + 2\varepsilon, \end{aligned}$$

as desired.

Define a randomized stopping time ϕ as follows:

$$\phi(r, \cdot) = \begin{cases} \mu + r\delta & \{\mu < \tau\} \cup A_0, \\ \tau & \{\mu = \tau\} \cap (A_1 \cup A_2), \\ \mu + r\delta & \{\mu = \tau\} \cap A_3. \end{cases}$$

The randomized stopping times ϕ and ϕ^* differ only over the set $\{\mu = \tau\} \cap (A_1 \cup A_2)$. Since over this set the payoff in Γ^* is w, provided the game terminates after time τ regardless of what the players play, and by (C2),

$$\inf_\nu \gamma_{\Gamma^*}(\phi, \nu) \geq V - 3\varepsilon. \tag{6}$$

Let ν be an arbitrary stopping time. Define a partition (B_0, B_1, B_2) of $[0,1] \times \Omega$ by

$$\begin{aligned} B_0 &= \{\mu + \delta < \tau\} \cup \{\nu < \tau\}, \\ B_1 &= \{\mu < \tau < \mu + \delta\} \cap \{\nu \geq \tau\}, \\ \text{and } B_2 &= \{\mu = \tau\} \cap \{\nu \geq \tau\}. \end{aligned}$$

Over B_0 the game terminates before time τ under (ϕ, ν). In particular,

$$\gamma_\Gamma(\phi, \nu; B_0) = \gamma_{\Gamma^*}(\phi, \nu; B_0). \tag{7}$$

By (C2), $P(B_1) < \varepsilon/M$, so that

$$\gamma_\Gamma(\phi, \nu; B_1) \geq \gamma_{\Gamma^*}(\phi, \nu; B_1) - 2\varepsilon. \tag{8}$$

Over $B_2 \cap A_0$ the game never terminates under (ϕ, ν), so that

$$\gamma_\Gamma(\phi, \nu; B_2 \cap A_0) = \gamma_{\Gamma^*}(\phi, \nu; B_2 \cap A_0) = 0. \tag{9}$$

Over $A_1 \cup A_2$, $\min\{a_\tau, c_\tau\} \geq w$, so that

$$\begin{aligned}\gamma_\Gamma(\phi,\nu;B_2\cap(A_1\cup A_2)) &= \mathbf{E}_{\lambda\otimes P}[\mathbf{1}_{B_2\cap(A_1\cup A_2)}(a_\tau\mathbf{1}_{\{\tau<\nu\}}+c_\tau\mathbf{1}_{\{\tau=\nu\}})] \\ &\geq \mathbf{E}_{\lambda\otimes P}[w\mathbf{1}_{\{\tau\leq\nu\}\cap B_2\cap(A_1\cup A_2)}] \\ &= \gamma_{\Gamma^*}(\phi,\nu;B_2\cap(A_1\cup A_2)).\end{aligned} \tag{10}$$

Finally, since $\lambda \otimes P(\{\mu + r\delta = \nu\}) = 0$, since $\{\mu = \tau\}$ on B_2, by Corollary 5, since $(c_t)_{t\geq 0}$ is progressively measurable, and since $a_\tau \geq b_\tau = w$ on A_3,

$$\begin{aligned}\gamma_\Gamma(\phi,\nu;B_2\cap A_3) &= \mathbf{E}_{\lambda\otimes P}[\mathbf{1}_{B_2\cap A_3}(a_{\mu+r\delta}\mathbf{1}_{\{\mu+r\delta<\nu\}}+b_\nu\mathbf{1}_{\{\mu+r\delta>\nu\}}+c_\nu\mathbf{1}_{\{\mu+r\delta=\nu\}})] \\ &= \mathbf{E}_{\lambda\otimes P}[\mathbf{1}_{B_2\cap A_3}(a_{\mu+r\delta}\mathbf{1}_{\{\mu+r\delta<\nu\}}+b_\nu\mathbf{1}_{\{\mu+r\delta>\nu\}})] \\ &\geq \mathbf{E}_{\lambda\otimes P}[\mathbf{1}_{B_2\cap A_3}(a_\tau\mathbf{1}_{\{\mu+r\delta<\nu\}}+b_\tau\mathbf{1}_{\{\mu+r\delta>\nu\}})]-2\varepsilon \\ &\geq \mathbf{E}_{\lambda\otimes P}[w\mathbf{1}_{B_2\cap A_3}]-2\varepsilon \\ &= \gamma_{\Gamma^*}(\phi,\nu;B_2\cap A_3)-2\varepsilon.\end{aligned} \tag{11}$$

Summing (7)–(11) and using (6) gives us

$$V - 3\varepsilon \leq \gamma_{\Gamma^*}(\phi,\nu) \leq \gamma_\Gamma(\phi,\nu) + 4\varepsilon,$$

as desired.

3.4. On final payoff. Our convention is that the payoff is 0 if no player ever stops. In fact, one can add a final payoff as follows. Let χ be an $\mathcal{A}$-measurable and integrable function. The expected payoff that corresponds to a pair of pure strategies (μ,ν) is

$$\mathbf{E}_P[a_\mu\mathbf{1}_{\{\mu<\nu\}}+b_\nu\mathbf{1}_{\{\mu>\nu\}}+c_\mu\mathbf{1}_{\{\mu=\nu<+\infty\}}+\chi\mathbf{1}_{\{\mu=\nu=+\infty\}}].$$

The expected payoff can be written as

$$\begin{aligned}\mathbf{E}_P[\chi]+\mathbf{E}_P\Big[&\Big(a_\mu-\mathbf{E}_P^{\mathcal{F}_\mu}[\chi]\Big)\mathbf{1}_{\{\mu<\nu\}}+\Big(b_\nu-\mathbf{E}_P^{\mathcal{F}_\nu}[\chi]\Big)\mathbf{1}_{\{\mu>\nu\}} \\ &+\Big(c_\mu-\mathbf{E}_P^{\mathcal{F}_\mu}[\chi]\Big)\mathbf{1}_{\{\mu=\nu<+\infty\}}\Big],\end{aligned}$$

where $\mathbf{E}_P^{\mathcal{F}_\mu}[\chi]$ is the conditional expectation of χ given the σ-algebra $\mathcal{F}_\mu$.

Define a process $d_t := \mathbf{E}_P^{\mathcal{F}_t}[\chi]$. Since the filtration satisfies the "usual conditions," $(d_t)_{t\geq 0}$ is a right-continuous martingale (see, e.g., Dellacherie and Meyer (1980, section VI-4) or Lepeltier and Maingueneau (1984, Theorem 4)). Hence we are reduced to the study of the standard stopping game $\Gamma^* = (\Omega, \mathcal{A}, P, (\mathcal{F}_t)_{t\geq 0}, (a_t^*, b_t^*, c_t^*)_{t\geq 0})$ with $a_t^* = b_t - d_t$, $b_t^* = b_t - d_t$, and $c_t^* = c_t - d_t$.

3.5. Right-continuity of the payoff process. For every $s \geq 0$, let $\Gamma[s]$ be the stopping game that starts at time s. Formally, $\Gamma[s]$ is given by $(\Omega, \mathcal{A}, P, (\mathcal{F}'_t, a'_t, b'_t, c'_t)_{t\geq 0})$, where for every $t \geq 0$, $\mathcal{F}'_t = \mathcal{F}_{t+s}$, $a_t = a_{t+s}$, $b_t = b_{t+s}$, and $c_t = c_{t+s}$. Let v_s be the value of $\Gamma[s]$.

The next proposition states that if the payoff processes are right-continuous, the process $(v_t)_{t\geq 0}$ is right-continuous as well.

PROPOSITION 7. *If the processes $(a_t, b_t, c_t)_{t\geq 0}$ are right-continuous, then so is $(v_t)_{t\geq 0}$.*

Proof. For every $t \geq 0$, denote $\tau[t] = \inf\{t \geq s\colon a_s \geq b_s\}$ and define the sets $A_0[t], A_1[t], A_2[t]$, and $A_3[t]$ as in the proof of Theorem 3 with respect to $\tau[t]$. Set

$$w_t = a_{\tau[t]}\mathbf{1}_{A_1[t]} + c_{\tau[t]}\mathbf{A_2}[\mathbf{t}] + b_{\tau[t]}\mathbf{1}_{A_3[t]}.$$

Now fix $t \geq 0$. On $\{a_t < b_t\}$, one has $w_t = w_s$ for every $s > t$ sufficiently close to t, so that by Lepeltier and Maingueneau (1984, Theorem 9), the value is right-continuous on this set.

On $\{a_t > c_t > b_t\}$, one has $v_s = c_s$ for every $s \geq t$ sufficiently close to t, and by the right-continuity of $(c_t)_{t\geq 0}$ the same conclusion holds.

On $\{a_t = c_t \geq b_t\}$, one has $\tau[t] = 0$ and $v_t = a_t = c_t$. Moreover, for every $\varepsilon > 0$ and every $s > t$ sufficiently small, one has (i) $a_s > a_t - \varepsilon = v_t - \varepsilon$ and $c_s > c_t - \varepsilon = v_t - \varepsilon$, so that $v_s > v_t - \varepsilon$, and (ii) $b_s < b_t + \varepsilon \leq v_t + \varepsilon$ and $c_s < c_t + \varepsilon = v_t + \varepsilon$, so that $v_s < v_t + \varepsilon$. In particular, $(v_t)_{t\geq 0}$ is right-continuous at t on this set.

A similar argument shows the right-continuity of $(v_t)_{t\geq 0}$ in all of the remaining cases. $\square$

3.6. Noisy stochastic duels. As mentioned in the introduction, the right-continuity of the payoff process can be used to derive, by induction and proper definition of a final payoff, the existence of an equilibrium in a more general class of games, in which (i) each player has to act at most M times, and (ii) the payoff depends on the number of times each player acted, as well as on the exact times in which the players acted. That is, the game is given by a filtration $(\mathcal{F}_t)_{t\geq 0}$ and, for every $0 \leq n, m \leq M$, a right-continuous process $u_{m,n}(t_1, \ldots, t_m, t'_1, \ldots, t'_n)$ that is defined whenever $t_1 < t_2 < \cdots < t_m$ and $t'_1 < t'_2 < \cdots < t'_n$, and such that $u_{m,n}(t_1, \ldots, t_m, t'_1, \ldots, t'_n)$ is $\mathcal{F}_{\max\{t_m, t'_n\}}$-measurable. If player 1 acts at times $t_1 < \cdots < t_m$ and player 2 acts at times $t'_1 < \cdots < t'_n$, with $0 \leq m, n \leq M$, the payoff is $u_{m,n}(t_1, \ldots, t_m, t'_1, \ldots, t'_n)$. This implies, in particular, that every noisy stochastic duel in which each player is endowed with finitely many bullets, the payoff is 1 if player 1 hits player 2, the payoff is –1 if player 2 hits player 1, and the accuracy process is right-continuous, admits a value.

Details are standard and omitted.

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