Operations Research Letters 48 (2020) 262–265

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

Logit equilibrium as an approximation of Nash equilibrium☆

Eilon Solan*, Omri N. Solan

The School of Mathematical Sciences, Tel Aviv University, Tel Aviv 6997800, Israel

ARTICLE INFO

Article history:
Received 18 November 2019
Received in revised form 6 March 2020
Accepted 24 March 2020
Available online 6 April 2020

Keywords:
Nash equilibrium
Logit equilibrium
Smooth manifold

ABSTRACT

We prove that the graph of the logit equilibrium correspondence is a smooth manifold, which is homeomorphic to the space of payoff functions and uniformly approximates the graph of the Nash equilibrium manifold.

1. Introduction

Theorem 1 in [3] shows that the graph of the Nash equilibrium correspondence is homeomorphic to the set of payoff functions. Proposition 2 in [5] proves that the graph of the Nash equilibrium correspondence can be uniformly approximated by a smooth manifold. In this note we provide a specific smooth manifold that is homeomorphic to the set of payoff functions and uniformly approximates the graph of the Nash equilibrium correspondence, namely, the graph of the logit equilibrium correspondence, a solution concept that was defined in [4] and sheds light on certain experimental results.

We now describe two applications of this result. [1,2] use the structure theorem of [3] to construct a homotopy-based method for computing equilibria in normal-form games and stationary equilibria in discounted stochastic games, respectively. Since the homotopy path is in general piecewise differentiable and not everywhere differentiable, numerical path tracing is impaired. The everywhere differentiable homotopy path, which is induced by the graph of the logit equilibrium correspondence, allows standard numerical tracing methods based on numerical integration to approximate, to any degree, Nash equilibria of normal-form games and stationary equilibria in discounted stochastic games.

Another application of our result is to the study of uniform equilibrium in general quitting games, see [6]. Under some constraints on the payoff function of the game, it can be shown that a limit of fixed points of a certain function, whose domain is an approximation of the graph of the Nash equilibrium correspondence, is a stationary equilibrium of the general quitting game. Two properties that are required for this approach are that the approximation is smooth (so that a topological fixed point theorem can be applied) and that every point on the approximation assigns positive probability to all actions of all players. Since [5] does not guarantee the existence of a smooth approximation that satisfies the latter property, the result of [5] is not sufficient for this application.

2. The model and main result

A *strategic game form* is a pair (I, A) where $I = \{1, 2, \dots, d\}$ is a finite set of players and $A = \times_{i\in I} A_i$ is the Cartesian product of finite sets of *pure strategies* for the players. A *payoff function for player i* for the strategic game form (I, A) is a function $u_i : A \to \mathbf{R}$, and a *payoff function* is a collection $u = (u_i)_{i\in I}$ of payoff functions for the players. Consequently, the set of all payoff functions is equivalent to $\mathbf{R}^{|A|\times|I|}$. A triplet (I, A, u) where u is a payoff function for the strategic game form (I, A) is a *game*.

A *mixed strategy* for player i is a probability distribution $x_i \in \Delta(A_i)$, and a *mixed strategy profile* is a collection $x = (x_i)_{i\in I}$ of mixed strategies for the players. It follows that the set of all mixed strategy profiles is $X := \times_{i\in I} \Delta(A_i) \subset \mathbf{R}^{\sum_{i\in I}|A_i|}$. A payoff function u_i for player i is extended to a function from X to $\mathbf{R}$ in a multilinear fashion.

A mixed strategy profile $x \in X$ is a *Nash equilibrium* of the game (I, A, u) if $u_i(x) \geq u_i(a_i, x_{-i})$ for every player $i \in I$ and every pure strategy $a_i \in A_i$. When the strategic game form is fixed, the graph of the Nash equilibrium correspondence is the collection of all pairs of a payoff function and Nash equilibrium in the game induced by this payoff function.

☆ E. Solan acknowledges the support of the Israel Science Foundation, grant #217/17. We thank Steffen Eibelhäuser for a useful discussion.
* Corresponding author.
E-mail addresses: eilons@post.tau.ac.il (E. Solan), omrisola@post.tau.ac.il (O.N. Solan).

https://doi.org/10.1016/j.orl.2020.03.011

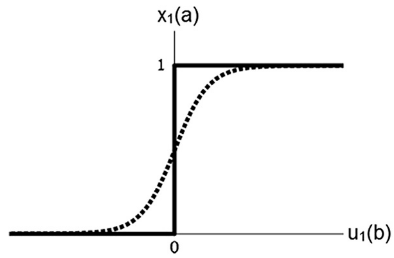

Fig. 1. One coordinate of the graphs of the Nash equilibrium correspondence (solid line) and logit equilibrium correspondence (dotted line) when $u_1(a) = 0$.

Definition 2.1. Let (I, A) be a strategic game form. The *graph of the Nash equilibrium correspondence* of (I, A) is the set

$$M := \left\{(u, x) \in \mathbf{R}^{|A|\times|I|} \times X : x \text{ is a Nash equilibrium of } (I, A, u)\right\} \subset \mathbf{R}^{|A|\times|I|} \times \mathbf{R}^{\sum_{i\in I}|A_i|}.$$

As mentioned above, [3] proved that the set M is homeomorphic to the set of games, namely, to $\mathbf{R}^{|A|\times|I|}$. An important concept that we will need is that of logit equilibrium, which we define now.

Definition 2.2 (*[4]*).Let (I, A, u) be a game and let $n > 0$. The mixed strategy profile x is a *logit equilibrium with parameter n* of the game (I, A, u) if for every player $i \in I$ and every pure strategy $a_i \in A_i$,

$$x_i(a_i) = \frac{\exp(nu_i(a_i, x_{-i}))}{\sum_{a'_i\in A_i}\exp(nu_i(a'_i, x_{-i}))}. \tag{1}$$

Standard continuity arguments show that a limit of logit equilibria with parameter n as n goes to infinity is a Nash equilibrium, see Theorem 2 in [4].

Definition 2.3. Let (I, A) be a strategic game form. For every real number n, the *graph of the logit equilibrium correspondence* of (I, A) is the set

$$M_n := \{(u, x) : x \text{ is a logit equilibrium with parameter } n \text{ of } (I, A, u)\} \subset \mathbf{R}^{|A|\times|I|} \times \mathbf{R}^{\sum_{i\in I}|A_i|}.$$

Example 2.4. Suppose that there is one player who has two actions: $I = \{1\}$ and $A_1 = \{a, b\}$. A payoff function for the player is then given by two real numbers $u_1(a)$ and $u_1(b)$. A mixed strategy $x_1 = (x_1(a), x_1(b))$ is a Nash equilibrium if

$$x_1(a) \in \begin{cases} \{1\} & x_1(a) > x_1(b), \\ \{0\} & x_1(a) < x_1(b), \\ [0,1] & x_1(a) = x_1(b). \end{cases}$$

A mixed strategy $x_1 = (x_1(a), x_1(b))$ is a logit equilibrium with parameter n if $x_1(a) = \exp(nu_1(a))/(\exp(nu_1(a)) + \exp(nu_1(b)))$. The section of the graphs of the Nash equilibrium and logit equilibrium correspondences when $u_1(a) = 0$ appear in Fig. 1. The larger n is, the closer the graph of the logit equilibrium correspondence gets to the graph of the Nash equilibrium correspondence.

Our first main result is that the graph of the logit correspondence is a smooth manifold.

Theorem 2.5. *The set M_n is a smooth manifold of dimension $|A| \times |I|$.*

Our second main result is that the graph of the logit correspondence uniformly approximates the graph of the Nash equilibrium correspondence.

Theorem 2.6. *There are a homeomorphism $\varphi : M \to \mathbf{R}^{|A|\times|I|}$ and smooth homeomorphisms $\varphi_n : M_n \to \mathbf{R}^{|A|\times|I|}$, $n > 0$, that satisfy the following property: For every $\varepsilon > 0$ there is $N = N(\varepsilon) > 0$ such that for every $n \geq N$ we have*

$$\|\varphi^{-1}(w) - (\varphi_n)^{-1}(w)\|_2 \leq \varepsilon, \quad \forall w \in \mathbf{R}^{|A|\times|I|}.$$

3. Proofs

To prove that M_n is a smooth manifold we need to study a certain function that will be used in the definition of the immersion between M_n and $\mathbf{R}^{|A|\times|I|}$. Recall that an *immersion* is a differentiable function between differentiable manifolds whose derivative is everywhere injective (one-to-one). The keen reader will identify the origin of this function and the proof of Theorem 2.5 in [3].

Lemma 3.1. *For every $n > 0$ define the function $g^{(n)} : \mathbf{R}^d \to \mathbf{R}^d$ by*

$$g_i^{(n)}(x) = x_i + \frac{\exp(nx_i)}{\sum_{j=1}^d \exp(nx_j)}, \quad \forall i \in \{1, 2, \ldots, d\}.$$

The function $g^{(n)}$ is one-to-one, onto, and an immersion.

Proof. Step 1: *The function $g^{(n)}$ is an immersion.*

An $n \times n$ matrix A is a *CL-matrix* if (a) its diagonal entries are positive, (b) its off-diagonal entries are negative, and (c) the sum of elements in each column is positive. Thus, CL-matrices are subclasses of both L-matrices and column strictly diagonally dominant matrices. By the Levy–Desplanques Theorem, every CL-matrix is invertible.

We first argue that the Jacobian matrix of $g^{(n)}$ is a CL-matrix at all points. Indeed, simple algebraic calculations show that for every $i \in \{1, 2, \ldots, d\}$,

$$\frac{\partial g_i^{(n)}}{\partial x_i}(x) = 1 + \frac{n\exp(nx_i)\left(\sum_{k\neq i}\exp(nx_k)\right)}{\left(\sum_{k=1}^d \exp(nx_k)\right)^2} > 0,$$

$$\frac{\partial g_i^{(n)}}{\partial x_j}(x) = -\frac{n\exp(n(x_i + x_j))}{\left(\sum_{k=1}^d \exp(nx_k)\right)^2} < 0, \quad \forall j \neq i.$$

In particular, Conditions (a) and (b) hold for the Jacobian matrix of $g^{(n)}$ at every point x. We also have

$$\sum_{i=1}^d g_i^{(n)}(x) = 1 + \sum_{i=1}^d x_i,$$

and therefore

$$\sum_{i=1}^d \frac{\partial g_i^{(n)}}{\partial x_j}(x) = 1 > 0, \quad \forall j \in \{1, 2, \ldots, d\},$$

so that Condition (c) holds as well, and the Jacobian matrix is a CL-matrix at all points. It follows that the Jacobian matrix is invertible at all points, hence $g^{(n)}$ is an immersion.

Step 2: *The function $g^{(n)}$ is onto.*

To prove that $g^{(n)}$ is onto we will show that its image is both open and closed. Since the Jacobian matrix of $g^{(n)}$ at every point x is invertible, by the Open Mapping Theorem the image of $g^{(n)}$ is an open set. To show that the image of $g^{(n)}$ is closed, note that $\|x - g^{(n)}(x)\|_2 \leq 1$ for every $x \in \mathbf{R}^d$, and consider a sequence

$(y^k)_{k\in\mathbf{N}}$ of points in the image of g that converges to a point y. For each $k \in \mathbf{N}$ let $x^k \in \mathbf{R}^d$ satisfy $y^k = g^{(n)}(x^k)$. Since $\|x^k - y^k\|_2 \le 1$, and since the sequence $(y^k)_{k\in\mathbf{N}}$ converges, it follows that there is a subsequence $(x^{k_l})_{l\in\mathbf{N}}$ that converges to a limit x. Since the function $g^{(n)}$ is continuous, $g^{(n)}(x) = y$, so that y is in the image of $g^{(n)}$, which implies that the image of $g^{(n)}$ is closed.

Step 3: *The function $g^{(n)}$ is one-to-one.*

We argue that any function whose Jacobian matrix is a CL-matrix is one-to-one. Indeed, let f be such a function, assume w.l.o.g. that $f(\vec{0}) = \vec{0}$, and fix $x \neq \vec{0}$. We will show that $f(x) \neq \vec{0}$. We have

$$f(x) = f(0) + \int_{t=0}^{1} df_{tx}\cdot x dt = \left(\int_{t=0}^{1} df_{tx} dt\right)\cdot x.$$

The matrix $\int_{t=0}^{1} df_{tx} dt$, as an integral of CL-matrices, is a CL-matrix, hence invertible. In particular, $f(x) = \left(\int_{t=0}^{1} df_{tx} dt\right)\cdot x \neq \vec{0}$, as claimed. ■

We are now ready to prove Theorem 2.5.

Proof of Theorem 2.5. [3] provided an equivalent representation for payoff functions. Let $u : A \to \mathbf{R}^{|I|}$ be a payoff function. For every $i \in I$ define two functions $\widetilde{u}_i : A \to \mathbf{R}$ and $\overline{u}_i : A_i \to \mathbf{R}$ by

$$\overline{u}_i(a_i) := \frac{1}{|A_{-i}|}\sum_{a_{-i}\in A_{-i}} u_i(a_i, a_{-i}),$$
$$\widetilde{u}_i(a) := u_i(a) - \overline{u}_i(a_i).$$

Denote $\langle \widetilde{u}, \overline{u}\rangle = (\widetilde{u}_i, \overline{u}_i)_{i\in I}$. Since $u_i(a) = \widetilde{u}_i(a) + \overline{u}_i(a_i)$, it follows that the mapping $u \mapsto \langle \widetilde{u}, \overline{u}\rangle$ is one-to-one and onto.

Fix $n > 0$ and define a function $z_n : M_n \to \mathbf{R}^{\sum_{i\in I}|A_i|}$ by

$$z_{n,i,a_i}(u, x) := u_i(a_i, x_{-i}) + \frac{\exp(nu_i(a_i, x_{-i}))}{\sum_{j\in I}\exp(nu_j(a_j, x_{-j}))}, \quad \forall i \in I, a_i \in A_i,$$

where the coordinates of every vector $\zeta \in \mathbf{R}^{\sum_{i\in I}|A_i|}$ are denoted $(\zeta_{i,a_i})_{i\in I, a_i\in A_i}$. Define now a function $\varphi_n : M_n \to \mathbf{R}^{|A|\times|I|}$ by $\varphi_n(u, x) := \langle \widetilde{u}, z_n(u, x)\rangle$. Lemma 3.1 implies that the function φ_n is one-to-one, onto, and an immersion. The result follows. ■

We now prove that the inverse of $g^{(n)}$ converges uniformly as n goes to infinity, and we provide an explicit form to the limit function, which is nothing but the homeomorphism defined by [3].

Lemma 3.2. *For every $n > 0$ let $h^{(n)} : \mathbf{R}^d \to \mathbf{R}^d$ be the inverse of $g^{(n)}$. Let $h : \mathbf{R}^d \to \mathbf{R}^d$ be the function defined by $h_i(y) := \min\{y_i, \alpha^*\}$ for every $i = 1, 2, \ldots, d$, where $\alpha^* := \max\left\{\alpha \in \mathbf{R}: \sum_{i=1}^d (y_i - \alpha)_+ = 1\right\}$. Then the sequence of functions $(h^{(n)})_{n>0}$ converges uniformly to the function h.*

Proof. Fix $\varepsilon > 0$, and let $n > 0$ be sufficiently large so that $\varepsilon > 1/(1 + \exp(\varepsilon n))$. Fix $y \in \mathbf{R}^d$ and define $x := h(y)$ and $x^{(n)} := h^{(n)}(y)$. Assume w.l.o.g. that $y_1 \le y_2 \le \cdots \le y_d$. By the definition of $g^{(n)}$ we have $x_1^{(n)} \le x_2^{(n)} \le \cdots \le x_d^{(n)}$. By the definition of h we have $x_1 \le x_2 \le \cdots \le x_d$. Since

$$\sum_{i=1}^d (y_i - \alpha^*)_+ = 1 = \sum_{i=1}^d (y_i - x_i^{(n)}) = \sum_{i=1}^d (y_i - x_i^{(n)})_+,$$

and since $x_1^{(n)} \le x_2^{(n)} \le \cdots \le x_d^{(n)}$, it follows that $x_d^{(n)} \ge \alpha^* = x_d$.

For every $i \in \{1, 2, \ldots, d\}$ denote $\alpha_i := y_i - x_i \ge 0$, and $\alpha_i^{(n)} := y_i - x_i^{(n)} \ge 0$. We now claim that $\alpha_i^{(n)} < \alpha_i + \varepsilon$. Indeed, assume to the contrary that for some $i \in \{1, 2, \ldots, d\}$ we have $\alpha_i^{(n)} \ge \alpha_i + \varepsilon$. Then in particular

$$x_i^{(n)} = y_i - \alpha_i^{(n)} \le y_i - \alpha_i - \varepsilon = x_i - \varepsilon \le x_d - \varepsilon \le x_d^{(n)} - \varepsilon.$$

Therefore, by the definition of $g^{(n)}$,

$$\begin{aligned}\varepsilon \le \alpha_i^{(n)} &= \frac{\exp(nx_i^{(n)})}{\sum_{j=1}^d \exp(nx_j^{(n)})}\\ &\le \frac{\exp(nx_i^{(n)})}{\exp(nx_i^{(n)} + nx_d^{(n)})}\\ &= \frac{1}{1 + \exp\big(n(x_d^{(n)} - x_i^{(n)})\big)} \le \frac{1}{1 + \exp(\varepsilon n)},\end{aligned}$$

a contradiction to the choice of n. Since $\sum_{i=1}^d \alpha_i^{(n)} = 1 = \sum_{i=1}^d \alpha_i$, we deduce that for every $i \in \{1, 2, \ldots, d\}$ we have $\alpha_i - d\varepsilon < \alpha_i^{(n)} < \alpha_i + \varepsilon$, which implies that $\|h^{(n)}(y) - h(y)\|_\infty \le d\varepsilon$, and the desired result follows. ■

Proof of Theorem 2.6. In the proof of Theorem 1 in [3] it was shown that the following function $\varphi : M \to \mathbf{R}^{|A|\times|I|}$ is a homeomorphism:

$$\varphi(u, x) := \langle \widetilde{u}, z(u, x)\rangle, \quad \forall (u, x) \in M,$$

where notations follow the proof of Theorem 2.5 and

$$z_{i,a_i}(u, x) := u_i(a_i, x_{-i}) + x_i(a_i), \quad \forall i \in I, a_i \in A_i.$$

Theorem 2.6 follows from Lemma 3.2. ■

Remark 3.3. A natural question is whether one can identify other smooth manifolds that uniformly approximate the graph of the Nash equilibrium correspondence. In this remark we address this question. Let (I, A) be a strategic game form, let u be a payoff function for the strategic form game (I, A), and let $f : \mathbf{R}^{|A|\times|I|} \times X \to X$. A mixed strategy profile x is an *f-equilibrium* for the game (I, A, u) if $x = f(u, x)$. The graph of the f-equilibrium correspondence is the set

$$M_f := \left\{(u, x) \in \mathbf{R}^{|A|\times|I|} \times X : x \text{ is an } f\text{-equilibrium of } (I, A, u)\right\}.$$

The concept of logit equilibrium with parameter n coincides with the concept of f-equilibrium, when $f_{i,a_i}(u, x)$ is given by the right-hand side of Eq. (1), for every $i \in I$, $a_i \in A_i$, and $(u, x) \in \mathbf{R}^{|A|\times|I|} \times X$.

Let now $(f^{(n)})_{n\ge 0}$ be a parameterized family of functions from $\mathbf{R}^{|A|\times|I|} \times X$ to X. Following closely the proofs above reveals sufficient conditions that ensure the validity of Theorems 2.5 and 2.6 w.r.t. the sets $(M_{f^{(n)}})_{n\ge 0}$ instead of the sets $(M_n)_{n\ge 0}$. Specifically, these two theorems hold as soon as the following conditions hold for every $n \ge 0$.

- The function $f^{(n)}$ is smooth on the relative interior of $\mathbf{R}^{|A|\times|I|} \times X$.
- $1 + \frac{\partial f_{i,a_i}^{(n)}}{\partial x_i(a_i)}(u, x) > 0$ and $\frac{\partial f_{i,a_i}^{(n)}}{\partial x_{i'}(a_{i'})}(u, x) < 0$ for every $(u, x) \in \mathbf{R}^{|A|\times|I|} \times X$, $i, i' \in I$, $a_i \in A_i$, and $a'_{i'} \in A_{i'}$, provided $(i, a_i) \neq (i', a'_{i'})$.
- $u_i(a_i, x_{-i}) \ge u_i(a'_i, x_{-i})$ if and only if $f_{i,a_i}^{(n)}(u, x) \ge f_{i,a'_i}^{(n)}(u, x)$, for every $(u, x) \in \mathbf{R}^{|A|\times|I|} \times X$, $i \in I$, and $a_i, a'_i \in A_i$.

Furthermore, for every $\varepsilon > 0$, $(u, x) \in \mathbf{R}^{|A|\times|I|} \times X$, $i \in I$, and $a_i, a'_i \in A_i$ the following holds:

- If $u_i(a_i, x_{-i}) \geq u_i(a'_i, x_{-i}) + \varepsilon$, then $\lim_{n\to\infty} f^{(n)}_{i,a'_i}(u, x) = 0$. Moreover, the convergence is uniform over $(u, x) \in \mathbf{R}^{|A|\times|I|} \times X$.

References

[1] S. Govindan, R. Wilson, A global Newton method to compute Nash equilibria, J. Econom. Theory 110 (1) (2003) 65–86.

[2] S. Govindan, R. Wilson, Global Newton method for stochastic games, J. Econom. Theory 144 (2009) 414–421.

[3] E. Kohlberg, J.-F. Mertens, On the strategic stability of equilibria, Econometrica 54 (5) (1986) 1003–1037.

[4] R.D. McKelvey, T.R. Palfrey, Quantal response equilibria for normal form games, Games Econom. Behav. 10 (1) (1995) 6–38.

[5] K. Ritzberger, The theory of normal form games from the differentiable viewpoint, Internat. J. Game Theory 23 (1994) 207–236.

[6] E. Solan, O.N. Solan, Sunspot equilibrium in general quitting games, 2019, arXiv:1803.00878.